# Homework 5

1. (a) For each of (a)–(e) in #1 of Homework 4 that you completed last week, determine which subgroups are normal subgroups.

   (b) For those of (a)–(e) that are normal subgroups, determine the quotient group by writing its Cayley table. Note that the entries of the table will be cosets.
   (If you do this right, it is not so much writing. The quotient groups are small.)

(a) 1(a) – 1(d) had left cosets = right cosets so are normal subgroups
1(e) did not and so is not a normal subgroup

(b) 1(a)

| | $\langle 5\rangle$ | $1+\langle 5\rangle$ | $2+\langle 5\rangle$ | $3+\langle 5\rangle$ | $4+\langle 5\rangle$ |
|---|---|---|---|---|---|
| $\langle 5\rangle$ | $\langle 5\rangle$ | $1+\langle 5\rangle$ | $2+\langle 5\rangle$ | $3+\langle 5\rangle$ | $4+\langle 5\rangle$ |
| $1+\langle 5\rangle$ | $1+\langle 5\rangle$ | $2+\langle 5\rangle$ | $3+\langle 5\rangle$ | $4+\langle 5\rangle$ | $\langle 5\rangle$ |
| $2+\langle 5\rangle$ | $2+\langle 5\rangle$ | $3+\langle 5\rangle$ | $4+\langle 5\rangle$ | $\langle 5\rangle$ | $1+\langle 5\rangle$ |
| $3+\langle 5\rangle$ | $3+\langle 5\rangle$ | $4+\langle 5\rangle$ | $\langle 5\rangle$ | $1+\langle 5\rangle$ | $2+\langle 5\rangle$ |
| $4+\langle 5\rangle$ | $4+\langle 5\rangle$ | $\langle 5\rangle$ | $1+\langle 5\rangle$ | $2+\langle 5\rangle$ | $3+\langle 5\rangle$ |

1(b)

| | $4\mathbb{Z}$ | $1+4\mathbb{Z}$ | $2+4\mathbb{Z}$ | $3+4\mathbb{Z}$ |
|---|---|---|---|---|
| $4\mathbb{Z}$ | $4\mathbb{Z}$ | $1+4\mathbb{Z}$ | $2+4\mathbb{Z}$ | $3+4\mathbb{Z}$ |
| $1+4\mathbb{Z}$ | $1+4\mathbb{Z}$ | $2+4\mathbb{Z}$ | $3+4\mathbb{Z}$ | $4\mathbb{Z}$ |
| $2+4\mathbb{Z}$ | $2+4\mathbb{Z}$ | $3+4\mathbb{Z}$ | $4\mathbb{Z}$ | $1+4\mathbb{Z}$ |
| $3+4\mathbb{Z}$ | $3+4\mathbb{Z}$ | $4\mathbb{Z}$ | $1+4\mathbb{Z}$ | $2+4\mathbb{Z}$ |

1(c)

| | $A_3$ | $(12)A_3$ |
|---|---|---|
| $A_3$ | $A_3$ | $(12)A_3$ |
| $(12)A_3$ | $(12)A_3$ | $A_3$ |

1(d)

| | $H$ | $(123)H$ | $(132)H$ |
|---|---|---|---|
| $H$ | $H$ | $(123)H$ | $(132)H$ |
| $(123)H$ | $(123)H$ | $(132)H$ | $H$ |
| $(132)H$ | $(132)H$ | $H$ | $(123)H$ |

2. Let $G_1$ and $G_2$ be groups, and define the set $G_1 \times G_2 = \{(a_1, a_2) : a_1 \in G_1, a_2 \in G_2\}$.

   (a) Define a binary operation on $G_1 \times G_2$ and show it makes $G_1 \times G_2$ into a group.

   (b) Show $H_2 = \{(e_1, a_2) : a_2 \in G_2\}$ is a subgroup of $G_1 \times G_2$.[1] Is it normal?

   (c) Explain how the quotient group $(G_1 \times G_2)/H_2$ is related to $G_1$.

(a) Define: for $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$

$$(a_1, a_2) \cdot (b_1, b_2) = (a_1 a_2, b_1 b_2).$$

Associative: Let $(a_1, a_2), (b_1, b_2), (c_1, c_2) \in G_1 \times G_2$.

Then $((a_1, a_2) \cdot (b_1, b_2)) \cdot (c_1, c_2) = (a_1 b_1, a_2 b_2) \cdot (c_1, c_2)$

$= ((a_1 b_1) c_1, (a_2 b_2) c_2) = (a_1 (b_1 c_1), a_2 (b_2 c_2))$

$= (a_1, a_2) \cdot (b_1 c_1, b_2 c_2) = (a_1, a_2) \cdot ((b_1, b_2) \cdot (c_1, c_2))$.

Identity: $e = (e_1, e_2)$.

$e \cdot (a_1, a_2) = (e_1, e_2) \cdot (a_1, a_2) = (e_1 a_1, e_2 a_2) = (a_1, a_2)$

for all $a_1 \in G_1$, $a_2 \in G_2$. Similarly $(a_1, a_2) \cdot e = (a_1, a_2)$.

Inverse: For $(a_1, a_2) \in G_1 \times G_2$ an inverse is

$(a_1^{-1}, a_2^{-1})$: $(a_1^{-1}, a_2^{-1}) \cdot (a_1, a_2) = (a_1^{-1} a_1, a_2^{-1} a_2) = (e_1, e_2)$

and similarly $(a_1, a_2) \cdot (a_1^{-1}, a_2^{-1}) = e$.

Therefore $G_1 \times G_2$ is a group with this operation.

[1] Here $e_1$ is the identity in $G_1$.

(b) $H_2 = \{(e_1, a_2) : a_2 \in G_2\} \subseteq G_1 \times G_2$

Let $(e_1, a_2)$, $(e_1, b_2) \in H_2$.

Then $(e_1, a_2)\cdot(e_1, b_2) = (e_1 e_1, a_2 b_2) = (e_1, a_2 b_2) \in H_2$.

Further, $e = (e_1, e_1) \in H$ and $(e_1, a_1)^{-1} = (e_1, a_1^{-1}) \in H_2$.

Thus $H_2$ is a subgroup of $G_1 \times G_2$.

It is also normal:

Let $(a_1, a_2) \in G_1 \times G_2$. Then for $(e_1, b_2) \in H_2$:

$$(a_1, a_2)(e_1, b_2)(a_1, a_2)^{-1} = (a_1, a_2)(e_1, b_2)(a_1^{-1}, a_2^{-1})$$
$$= (a_1 e_1 a_1^{-1}, a_2 b_2 a_2^{-1}) = (e_1, a_2 b_2 a_2^{-1}) \in H_2.$$

Thus $aH_2a^{-1} \subseteq H_2$ for all $a \in G_1 \times G_2 \Rightarrow H_2$ normal.

(c) Consider the map $\phi : G_1 \to G_1 \times G_2 / H_2$

given by $\phi(a_1) = (a_1, e_2)H_2$.

This is onto: any coset $(a_1, b_1)H_2 = (a_1, e_2)H_2 = \phi(a_1)$

It is 1-1: $\phi(a_1) = \phi(b_1)$ implies (look at the sets!)

$(a_1, e_2)H_2 = (b_1, e_2)H_2 \Rightarrow a_1 = b_1$

Thus $\phi$ is a 1-1, onto map. It also respects group operations: $\phi(a_1 b_1) = (a_1 b_1, e_2)H_2 = \phi(a_1)\phi(b_1)$.

($\phi$ is an isomorphism.)

3. Let $G$ be a group and $N \subset G$ a normal subgroup.

   (a) If $G$ is abelian, show that $G/N$ is abelian. Is the converse true? Prove it or give a counterexample.

   (b) Is it true that if $N$ and $G/N$ are cyclic, then $G$ is cyclic? Prove it or give a counterexample.

(a) $G$ abelian $N \subseteq G$ normal

Let $aN, bN \in G/N$. Then $(aN)(bN) = abN = baN$

(ab=ba since G is abelian) $= (bN)(aN)$.

Thus $G/N$ is abelian.

If $G/N$ is abelian, it is not nec. true $G$ is abelian.

For example take $A_3 = \{e, (123), (132)\} \subseteq S_3$

Then $S_3/A_3$ is abelian of order 2.

But of course $S_3$ is not abelian.

(b) This is not true.

Use the same example.

$S_3/A_3$ is cyclic of order 2 and

$A_3$ is cyclic of order 3

but of course $S_3$ is not cyclic.

(An example where $G$ is abelian:

$G = \mathbb{Z}_2 \times \mathbb{Z}_2$ $\quad N = \{0\} \times \mathbb{Z}_2$

Then $N$ and $G/N$ cyclic order 2, $G$ not cyclic)

4. Consider the group $G \subset \mathrm{GL}_2(\mathbb{R})$ consisting of upper triangular matrices:

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in \mathbb{R}, \ ac \neq 0 \right\}$$

Let $H \subset G$ be the subset of $G$ consisting of those matrices in $G$ with $a = c = 1$.

(a) Show $H$ is an abelian subgroup of $G$.

(b) Show $H$ is normal in $G$.

(c) Show $G/H$ is abelian.

(a) $H = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$ Let $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in H$.

Then $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} \in H$. Also $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in H$ (take $a=0$)

and $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \in H$. So $H$ is a subgroup.

Finally $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

So $H$ is abelian.

(b) We aim to show $AHA^{-1} \subseteq H$ for all $A \in G$.

Let $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in H$, $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Then

$$A\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}A^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \cdot \frac{1}{ac}$$

$$= \begin{pmatrix} a & ax+b \\ 0 & c \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}\frac{1}{ac}$$

$$= \begin{pmatrix} ac & -ab + a^2x + ab \\ 0 & ac \end{pmatrix}\frac{1}{ac} = \begin{pmatrix} 1 & ax/c \\ 0 & 1 \end{pmatrix} \in H$$

Thus $H$ is normal in $G$.

(c) $G/H$ abelian iff for all $A, A' \in G$ we have

$$(AH)(A'H) = AA'H = A'AH = (A'H)(AH)$$

$$\Leftrightarrow \quad AA'A^{-1}(A')^{-1}H = H \quad \Leftrightarrow \quad AA'A^{-1}(A')^{-1} \in H.$$

Write $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $A' = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$.

Then $A^{-1} = \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \frac{1}{ac}$, $A' = \begin{pmatrix} c' & -b' \\ 0 & a' \end{pmatrix} \frac{1}{a'c'}$.

We compute

$$AA'A^{-1}(A')^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}\begin{pmatrix} c' & -b' \\ 0 & a' \end{pmatrix} \frac{1}{aca'c'}$$

$$= \begin{pmatrix} aa' & ab'+bc' \\ 0 & cc' \end{pmatrix}\begin{pmatrix} cc' & -cb'-ba' \\ 0 & aa' \end{pmatrix} \frac{1}{aca'c'}$$

$$= \begin{pmatrix} aa'cc' & * \\ 0 & cc'aa' \end{pmatrix} \frac{1}{aca'c'}$$

$$= \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \in H$$

where $*$ = some real number we don't care about.

Thus $AA'A^{-1}(A')^{-1} \in H$ for all $A, A' \in G \Rightarrow G/H$ abelian.