Homework 4

1. List the left and right cosets of the subgroups in the following list.
   (a) The subgroup $\langle 5 \rangle$, generated by 5 (mod 20), inside $(\mathbb{Z}_{20}, +)$.
   (b) The subgroup $4\mathbb{Z} = \{4k : k \in \mathbb{Z}\}$ inside the group $(\mathbb{Z}, +)$.
   (c) The subgroup A_3 inside the symmetric group S_3.
   (d) The subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ in the group A_4.
   (e) The subgroup $H = \{e, (123), (132)\}$ in the group A_4.

   For which of these examples does it happen that every right coset is a left coset, and every left coset is a right coset?

2. Let G be a group and $H \subset G$ a subgroup with index 2, i.e. $[G : H] = 2$. Show that $aH = Ha$ for all $a \in G$.

3. Recall that $\mathrm{GL}_2(\mathbb{R})$ is the group of real 2×2 matrices with non-zero determinant, and $\mathrm{SL}_2(\mathbb{R})$ is the subgroup of those matrices with determinant 1. Describe the right cosets of $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{GL}_2(\mathbb{R})$, and find the index of this subgroup.

4. Use Euler's Theorem or Fermat's Little Theorem to help compute the following.
   (a) 7^{26} (mod 15)
   (b) The last digit of 97^{123} (Hint: pass to integers mod 10)
   (c) 15^{83} (mod 41)

5. Suppose G is a finite group, and $a \in G$. Suppose n is an integer greater than 1 that divides the order of G. Show that a^n cannot generate G, i.e. $\langle a^n \rangle \neq G$.

6. Let G be a finite group of order pq where p and q are distinct primes. Show that if $a, b \in G$ are non-identity elements of different orders, then the only subgroup in G containing a and b is the whole group G.

7. Let G be a group. Given $a, b \in G$, we say a is *conjugate* to b if there exists $g \in G$ such that $a = gbg^{-1}$. Define $\sim$ as follows: $a \sim b$ if and only if a is conjugate to b.
   (a) Show that $\sim$ is an equivalence relation on G.
   (b) What are the equivalence classes of this relation if G is abelian?
   (c) Compute the equivalence classes of this relation for the groups S_3 and A_4.

8. Directly write down a bijection (where it sends all elements) from $\mathbb{Z}_{15}^\times$ to the Cartesian product $\mathbb{Z}_3^\times \times \mathbb{Z}_5^\times$. Is there a similar bijection from $\mathbb{Z}_{24}^\times$ to $\mathbb{Z}_4^\times \times \mathbb{Z}_6^\times$? Explain.