Homework 3

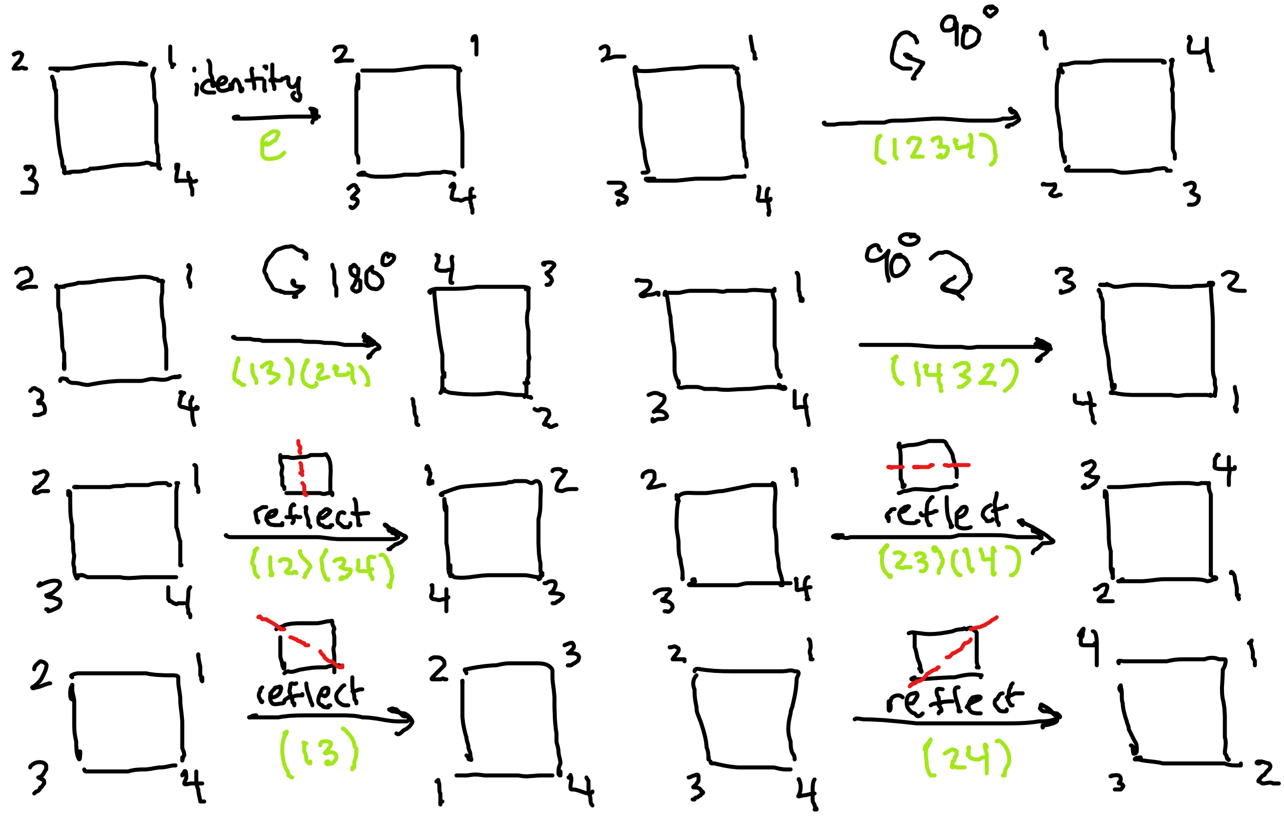
1. Let G be the group of symmetries of a square in the 2-dimensional plane.

   (a) Analogous to what we did for the symmetries of an equilateral triangle, write down all symmetries of the square. What is the order of the group G?

   (b) Label the 4 vertices of the square with the numbers $1, 2, 3, 4$. Accounting for how these labels are moved around by each symmetry, write down a subgroup $H \subset S_4$ which corresponds to G.

   (c) Is H all of S_4? Is it contained in the alternating group A_4?

(a)



There are 8 symmetries so $|G| = 8$.

(b)

$$H = \{e, (1234), (13)(24), (1432), (12)(34), (23)(14), (13), (24)\}$$

(c) H is not all of S_4 ($|H| = 8$ and $|S_4| = 24$).

H is not contained in A_4 because $(13) \in H$ is <u>odd</u>.

2. Compute the following compositions of permutations.

   (a) (1345)(234)

   (b) (143)(23)(24)

   (c) $(1354)^{100}$

(a) $(1345)(234) = (2\ 4)(1\ 3\ 5)$

$2 \mapsto 3$

$3 \mapsto 4$

$4 \mapsto 2$

$2 \mapsto 2$

$1 \mapsto 1$

$1 \mapsto 3$

$3 \mapsto 4$

$4 \mapsto 5$

$5 \mapsto 5$

$5 \mapsto 1$

(b) $(143)(23)(24) = (1\ 4\ 3)(2\ 4\ 3)$

$= (1\ 4)(2\ 3)$

(c) $(1\ 3\ 5\ 4)^2 = (1354)(1354) = (15)(34)$

$(1\ 3\ 5\ 4)^3 = (1354)(15)(34) = (1453)$

$(1354)^4 = (1354)(1453) = e$

$(1354)^{100} = ((1354)^4)^{25} = e^{25} = e.$

3. (a) What is $\operatorname{ord}(\sigma)$ for $\sigma \in S_n$ equal to a cycle of length l? Explain.
   (b) Recall that an arbitrary permutation $\sigma \in S_n$ can be written as $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$ where each σ_i is a cycle and they are all disjoint. Show that
   $$\operatorname{ord}(\sigma) = \operatorname{lcm}(l_1, \ldots, l_k)$$
   where l_i is the length of the cycle σ_i.
   (c) Write down all possible orders of elements in S_7.

(a) Let $\sigma = (a_1 \ldots a_\ell)$ be a cycle of length ℓ.

Then as a function $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ we have

$$\sigma(a_1) = a_2$$
$$\sigma^2(a_1) = \sigma(\sigma(a_1)) = \sigma(a_2) = a_3$$
$$\sigma^3(a_1) = a_4$$
$$\vdots$$
$$\sigma^{\ell-1}(a_1) = a_\ell$$

In particular, $\sigma^{\ell-1} \neq e$. On the other hand $\sigma^\ell(a_1) = \sigma(\sigma^{\ell-1}(a_1)) = \sigma(a_\ell) = a_1$ and similarly $\sigma^\ell(a_i) = a_i$ for all $i = 1, \ldots, \ell$.
Thus $\sigma^\ell = e$. It follows that $\operatorname{ord}(\sigma) = \ell$.

(b) $\sigma = \sigma_1 \cdots \sigma_k$, σ_i disjoint cycles of lengths ℓ_i

$L = \operatorname{lcm}(\ell_1, \ldots, \ell_k)$. disjoint cycles commute

$$\sigma^L = (\sigma_1 \cdots \sigma_k)^L = \sigma_1^L \cdots \sigma_k^L = \sigma_1^{m_1\ell_1} \cdots \sigma_k^{m_k\ell_k}$$
$$= (\sigma_1^{\ell_1})^{m_1} \cdots (\sigma_k^{\ell_k})^{m_k} = e^{m_1} \cdots e^{m_k} = e.$$

Thus $\operatorname{ord}(\sigma) \leq L$.

Suppose $\operatorname{ord}(\sigma) = N < L$. Then ℓ_i does not divide N for some $i = 1, \dots, k$. Suppose without loss of generality that $i = 1$.

Write $N = \ell_1 \cdot q + r$ where $0 < r < \ell_1$.

Since $\sigma_1^r \neq e$ (r is less than $\operatorname{ord}(\sigma_1) = \ell_1$) there is some $t \in \{1, 2, \dots, n\}$ such that $(\sigma_1^r)(t) \neq t$. As σ_i for $i \neq 1$ is disjoint from σ_1, we have

$$\sigma^N(t) = \sigma_1^N(t) = \sigma_1^r\Big(\underbrace{\overset{e}{\overset{\shortparallel}{\sigma_1^{\ell_1}}}(\dots(\overset{e}{\overset{\shortparallel}{\sigma_1^{\ell_1}}}}_{q \text{ times}}(t)\dots)\Big)$$

$$= \sigma_1^r(t) \neq t.$$

Thus $\sigma^N \neq e$. Finally, $\operatorname{ord}(\sigma) = L = \operatorname{lcm}(\ell_1, \dots, \ell_k)$.

(c) Possible orders of elements in S_7: $\underline{1} = \operatorname{ord}(e)$

$\underline{2}$ like (12) or $(12)(34)$ or $(12)(34)(56)$

$\underline{3}$ like (123) or $(123)(456)$

$\underline{4}$ like (1234) or $(1234)(56)$

$\underline{5}$ like (12345)

$\underline{6}$ like (123456) or $(12)(345)$

$\underline{7}$ like (1234567)

$\underline{10}$ like $(12)(34567)$

$\underline{12}$ like $(123)(4567)$

4. In lecture we saw that every cycle in S_n is a product of transpositions. Use this to explain how the parity of a cycle is determined by the length of the cycle.

Let $\sigma = (a_1 \dots a_\ell)$ be cycle of length ℓ.

Then
$$\sigma = (a_1\ a_\ell)(a_1\ a_{\ell-1}) \cdots (a_1\ a_3)(a_1\ a_2)$$

$$\left[\text{or} \quad \sigma = (a_3\ a_2)(a_4\ a_3) \cdots (a_k\ a_{k-1})(a_1\ a_k)\right]$$

In either case, σ is a product of $\ell - 1$ transpositions.

Thus we see that the parity of σ is the parity of $\ell - 1$, or:

$$\ell \text{ odd} \iff \sigma \text{ even}$$
$$\ell \text{ even} \iff \sigma \text{ odd}$$

5. Determine whether each permutation is even or odd.
   (a) (14356)
   (b) (156)(234)
   (c) (17254)(1423)(154632)

(a) Cycle of length 5
so it is even, by #4

(b) (1 5 6) (2 3 4) So the product is also even.

length 3 even
length 3 even

(c) (17254)(1423)(154632)

length 5 even
length 4 odd
length 6 odd

Thus the product is even.