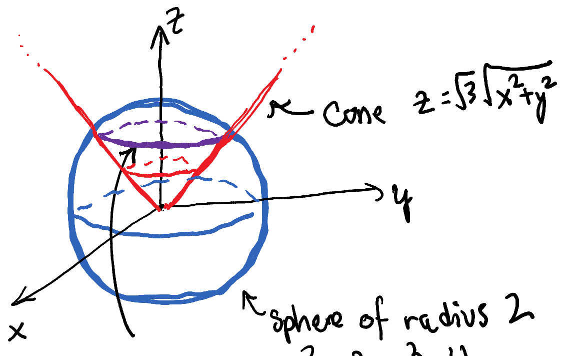
Practice Problems for Midterm 3 — Solutions

1. $f(x,y,z) = x + 2y + z$ $\quad R$ = solid given by $x^2+y^2+z^2 \le 4$
$\sqrt{3(x^2+y^2)} \le z$

Set up $\iiint_R f(x,y,z)\,dV$:

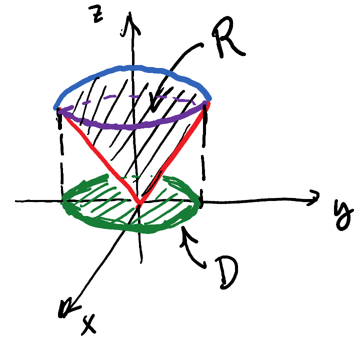
(a) in rectangular coordinates:

R is the solid region above the red cone and below the blue sphere. Compute the intersection of the surfaces.



intersection:
plug $z^2 = 3(x^2+y^2)$
into $x^2+y^2+z^2=4$
to get $x^2+y^2+3(x^2+y^2)=4$
$\longrightarrow x^2+y^2=1.$

Thus D is the unit disk $x^2+y^2 \le 1$



$$\iint_D \left(\int_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x+2y+z)\,dz \right) dA$$

$$= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} (x+2y+z)\,dz\,dy\,dx$$

(b) in cylindrical coordinates:
Make the substitutions indicated.

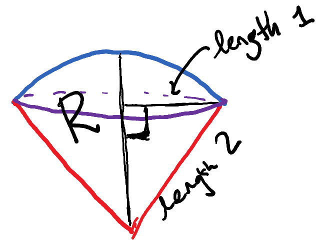
$x^2+y^2=r^2$
$x = r\cos\theta$
$y = r\sin\theta$
(z unchanged)

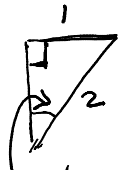
$dx\,dy\,dz \leftrightarrow r\,dr\,d\theta\,dz$

unit disk D $\quad x^2+y^2 \le 1 \quad \leftrightarrow \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$$\int_0^{2\pi} \int_0^{1} \int_{\sqrt{3}r}^{\sqrt{4-r^2}} (r\cos\theta + 2r\sin\theta + z)\, r\,dz\,dr\,d\theta.$$

(c) in spherical coordinates:





angle is $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$x^2+y^2+z^2=\rho^2$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Make these substitutions to get

R in spherical coords:

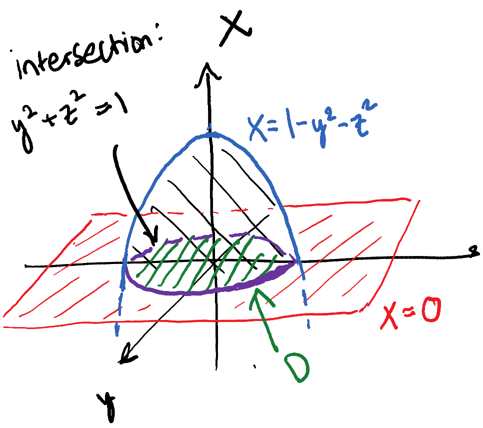
$0 \le \rho \le 2$

$0 \le \phi \le \pi/6$

$0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2} (\sin\phi\cos\theta + 2\sin\phi\sin\theta + \cos\phi)\,\rho^3 \, d\rho \, d\phi \, d\theta$$

2. Volume of the region between $x=0$ and $x=1-y^2-z^2$.

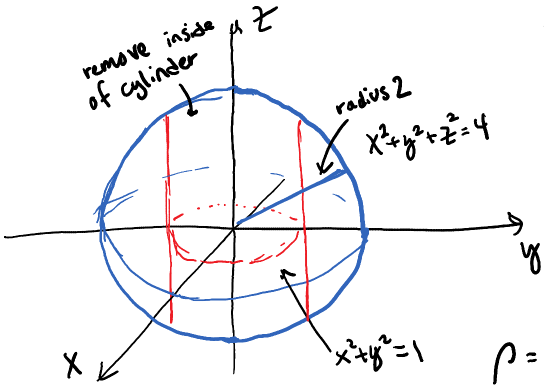


$$\text{Volume} = \iiint_R dV = \iint_D \left(\int_0^{1-y^2-z^2} dx \right) dy\,dz$$

$$= \iint_D (1-y^2-z^2)\,dy\,dz$$ use polar coords. $y = r\cos\theta$, $z = r\sin\theta$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Big|_{r=0}^{r=1} \cdot 2\pi = \frac{\pi}{2}$$

3.



$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2=r^2$

$dx\,dy\,dz \leftrightarrow r\,dr\,d\theta\,dz$

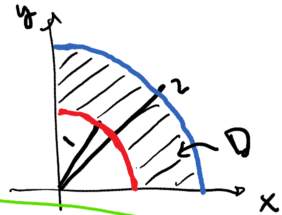
$\rho = z^2 + \sqrt{x^2+y^2} = z^2 + r$

We only want the portion in the 1st octant.

$0 \le z \le \sqrt{4-x^2-y^2} = \sqrt{4-r^2}$.

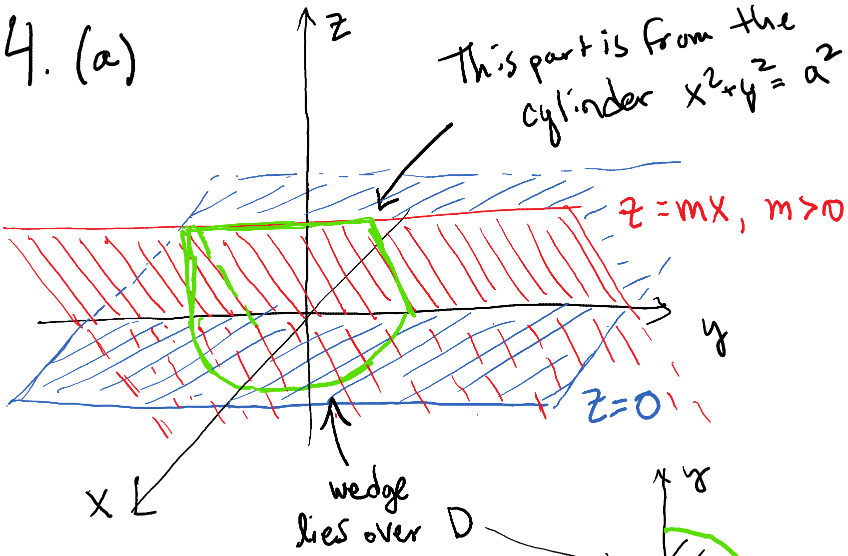
lies over the region D.

$1 \le r \le 2$, $0 \le \theta \le \pi/2$.



$$\int_0^{\pi/2}\int_1^2\int_0^{\sqrt{4-r^2}} (z^2+r)\,r\,dz\,dr\,d\theta.$$

4. (a)

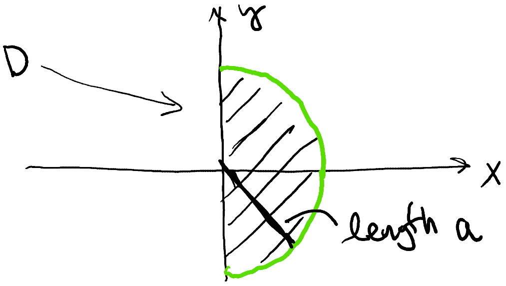


Volume of wedge

$$= \iiint_{\text{wedge}} dV$$

$$= \iint_D \left(\int_0^{mx} dz \right) dA$$

$$= \iint_D mx \cdot dA$$



in polar coordinates D is given by $0 \le r \le a$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Continuing, $\displaystyle\iint_D mx\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^a m r\cos\theta\, r\,dr\,d\theta = m \frac{r^3}{3}\Big|_{r=0}^{r=a} \cdot (-\sin\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} = \frac{2ma^3}{3}$

(b) Use spherical coordinates to evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2\,dz\,dy\,dx$$

The region is the part of the unit ball $x^2+y^2+z^2 \le 1$ in 1st octant

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$x^2+y^2+z^2=\rho^2$

thus $(x^2+y^2+z^2)^2 \to \rho^4$.

$dz\,dy\,dx \longleftrightarrow \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

This region is:

$0 \le \rho \le 1$
$0 \le \phi \le \pi/2$
$0 \le \theta \le \pi/2$

So we get $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \rho^6 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{\pi}{2}\cdot\left(\cos\phi\Big|_0^{\frac{\pi}{2}}\right)\cdot\left(\frac{\rho^7}{7}\Big|_0^1\right) = \frac{\pi}{14}$

5. $\vec{r}(t) = \langle \overset{x(t)}{\cos(t)}, \overset{y(t)}{\sin(t)}, \overset{z(t)}{4t} \rangle$, $0 \le t \le \pi$

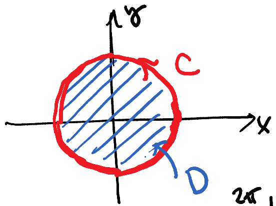
Then $\vec{r}\,'(t) = \langle -\sin(t), \cos(t), 4 \rangle$. $\vec{F} = \langle x, 0, z \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{\pi} \langle \cos(t), 0, 4t \rangle \cdot \langle -\sin(t), \cos(t), 4 \rangle\,dt$$

$$= \int_0^{\pi} (-\cos(t)\sin(t) + 16t)\,dt = \left(\frac{\cos^2(t)}{2} + 8t^2 \right)\Big|_0^{\pi} = 8\pi^2$$

either substitution (e.g. $u = \cos(t)$, $du = -\sin(t)\,dt$) or use $\sin(2t) = 2\sin(t)\cos(t)$.

6. $\vec{F} = \langle -yx^2 + e^{x^2}, xy^2 - e^{y^2} \rangle = \langle P, Q \rangle$.



Compute work along unit circle, counterclockwise.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy \underset{\text{Green's Theorem}}{=} \iint_D (Q_x - P_y)\,dx\,dy = \iint_D (y^2 + x^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta = 2\pi \cdot \frac{r^4}{4}\Big|_0^1 = \frac{\pi}{2}$$

7. $\vec{F} = \langle ye^{xy}, xe^{xy} - z^3, 2\sin(z) - 3yz^2 \rangle$

(a)

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ ye^{xy} & xe^{xy} - z^3 & 2\sin(z) - 3yz^2 \end{vmatrix} = \langle \cancel{3z^2} - \cancel{3z^2}, 0 - 0, \cancel{xye^{xy}} + \cancel{e^{xy}} - \cancel{yxe^{xy}} - \cancel{e^{xy}} \rangle = \vec{0}$$

(b) Find f such that $\nabla f = \vec{F}$: $f_x = ye^{xy}$, $f_y \overset{(*)}{=} xe^{xy} - z^3$, $f_z \overset{(**)}{=} 2\sin(z) - 3yz^2$

1) $f = \int f_x\,dx = \int ye^{xy}\,dx = \frac{1}{y} \cdot ye^{xy} + g(y,z) = e^{xy} + g(y,z)$.

2) $f_y = xe^{xy} + g_y \overset{(*)}{=} xe^{xy} - z^3 \longrightarrow g_y = -z^3 \longrightarrow g = \int g_y\,dy = -\int z^3\,dy = -yz^3 + h(z)$

3) $f_z = g_z = -3yz^2 + h' \overset{(**)}{=} 2\sin(z) - 3yz^2 \longrightarrow h' = 2\sin(z) \to h = \int h'\,dz = \int 2\sin(z)\,dz = -2\cos(z) + C$

constant can be anything

Thus $f(x,y,z) = e^{xy} - yz^3 - 2\cos(z) + C$

(c)

$$\text{work} = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} \overset{\text{Fundamental Theorem for Line Integrals}}{=} f(1,0,\pi) - f(0,5,0) = (\cancel{e^0} - \cancel{0} - 2(-1) + \cancel{C}) - (\cancel{e^0} - \cancel{0} - 2 + \cancel{C}) = 4$$

8. $\vec{F} = \langle z, x, y \rangle$

(a) work along straight line from $A = (3,0,0)$ to $B = (0, \frac{\pi}{2}, 3)$:

$\vec{r}(t) = \langle 3,0,0 \rangle + t\langle -3, \frac{\pi}{2}, 3 \rangle = \langle 3-3t, \frac{\pi}{2}t, 3t \rangle = \langle x(t), y(t), z(t) \rangle, \quad 0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 \langle 3t, 3-3t, \tfrac{\pi}{2}t \rangle \cdot \langle -3, \tfrac{\pi}{2}, 3 \rangle\, dt = \int_0^1 \left(-9t + \tfrac{\pi}{2}(3-\cancel{3t}) + \cancel{\tfrac{3\pi}{2}t}\right) dt$$

$$= \int_0^1 \left(\tfrac{3\pi}{2} - 9t\right) dt = \frac{3\pi}{2} - \frac{9}{2}.$$

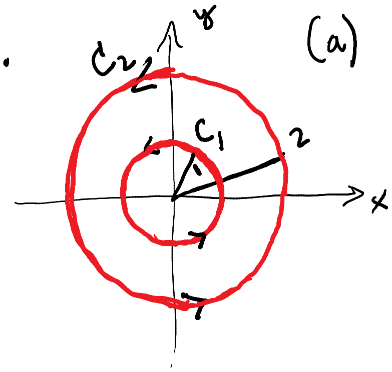
(b) use instead the helix $\vec{r}(t) = \langle 3\cos(t), t, 3\sin(t) \rangle, \quad 0 \le t \le \frac{\pi}{2}$. $\vec{r}\,'(t) = \langle -3\sin(t), 1, 3\cos(t) \rangle$

$$\text{work} = \int_0^{\pi/2} \langle 3\sin(t), 3\cos(t), t \rangle \cdot \langle -3\sin(t), 1, 3\cos(t) \rangle\, dt = \int_0^{\pi/2} \left(-9\sin^2(t) + 3\cos(t) + 3t\cos(t)\right) dt$$

$$= -9\int_0^{\pi/2} \frac{1-\cos(2t)}{2}\, dt + 3\sin(t)\Big|_0^{\pi/2} + 3t\sin(t)\Big|_0^{\pi/2} - 3\int_0^{\pi/2} \sin(t)\, dt = -\frac{9\pi}{4} + \cancel{3} + \frac{3\pi}{2} - \cancel{3} = \frac{-3\pi}{4}$$

$\vec{F}$ is not conservative. If it was, the Fundamental Theorem for line integrals would imply that the answers in (a) and (b) should be equal.

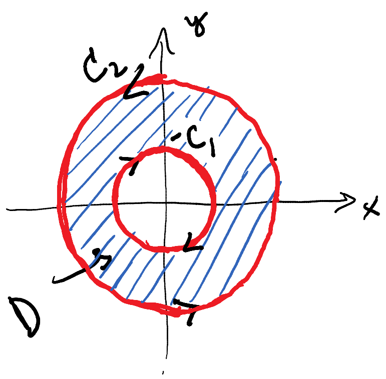
9.



(a) $\vec{F} = \langle P, Q \rangle$ Green's Theorem implies:

$$\int_D \operatorname{div} \vec{F}\, dx\, dy = \int_{C_1} \vec{F} \cdot \hat{n}\, ds = 10$$
(given).

here D is $x^2 + y^2 \le 1$



$$\int_{C_2} \vec{F} \cdot d\vec{r} - \underbrace{\int_{C_1} \vec{F} \cdot d\vec{r}}_{\substack{=17 \\ \text{given}}} = \iint_{\substack{D \\ \text{annulus}}} (\cancel{Q_x - P_y})\, dA = 0$$

given 0

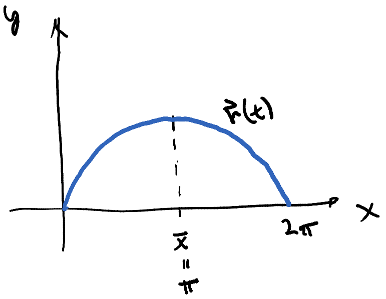
Thus $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r} = 17.$

10. First find the mass. $\vec{r}(t) = \langle t-\sin(t),\ 1-\cos(t)\rangle$, $0 \le t \le 2\pi$. $\rho = \rho(x,y) = 1$.

$$m = \text{mass} = \int_C \rho\, ds = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{(1-\cos(t))^2 + \sin^2(t)}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos(t)}\, dt$$

use: $\sin^2(t/2) = \dfrac{1-\cos(t)}{2}$.

$$= \int_0^{2\pi} 2\sin(t/2)\, dt = 4\cos(t/2)\Big|_0^{2\pi} = 8.$$



Center of mass: $(\bar{x}, \bar{y})$. By symmetry $\bar{x} = \pi$.

$$\bar{y} = \frac{1}{m}\int_0^{2\pi} y \cdot \rho\, ds = \frac{1}{8}\int_0^{2\pi} (1-\cos(t)) \cdot 1 \cdot 2\sin(t/2)\, dt$$

$$= \frac{1}{8}\int_0^{2\pi} 2\sin(t/2)\,dt - \frac{1}{4}\int_0^{2\pi} \cos(t)\sin(t/2)\, dt$$

use $\cos(t) = 2\cos^2(t/2) - 1$

$$= 1 - \frac{1}{4}\int_0^{2\pi} \left(2\cos^2(t/2) - 1\right)\sin(t/2)\, dt$$

$$= 1 - \frac{1}{2}\cos(t/2)\Big|_0^{2\pi} + \int_0^{2\pi} \cos^2(t/2)\, d\left(\cos(t/2)\right)$$

$$= 2 + \frac{\cos^3(t/2)}{3}\Big|_0^{2\pi} = 2 - \frac{2}{3} = \frac{4}{3}$$

Thus the center of mass is

$$(\bar{x}, \bar{y}) = (\pi, 4/3).$$