Practice Problems for Exam #2.

1) $f(x,y)$, $x = st$ $y = e^{st}$

a) $\frac{\partial x}{\partial t} = s$, $\frac{\partial y}{\partial t} = se^{st}$

b) $\frac{\partial f}{\partial t} \overset{\text{chain rule}}{=} \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = sf_x + se^{st} f_y$

c) $\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}\left(sf_x + se^{st} f_y \right) = s\frac{\partial}{\partial t}(f_x) + se^{st}\frac{\partial}{\partial t}(f_y) + s^2 e^{st} f_y$

$\overset{\text{chain rule}}{=} s\left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial t} \right) + se^{st}\left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial t} \right) + s^2 e^{st} f_y$

$= s(sf_{xx} + se^{st} f_{xy}) + se^{st}(sf_{xy} + se^{st} f_{yy}) + s^2 e^{st} f_y$

$= s^2 f_{xx} + 2s^2 e^{st} f_{xy} + s^2 e^{2st} f_{yy} + s^2 e^{st} f_y$

2) $f(x,y) = 3x^2 - xy + y^3$

a) $\nabla f = \langle 6x - y, 3y^2 - x \rangle$ $\nabla f(1,2) = \langle 4, 11 \rangle$

$\vec{v} = \langle 3, 4 \rangle$ $|\vec{v}| = \sqrt{9+16} = \sqrt{25} = 5$ unit vector: $\vec{u} = \langle \frac{3}{5}, \frac{4}{5} \rangle$

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u} = \langle 4, 11 \rangle \cdot \langle 3/5, 4/5 \rangle$

$= \frac{12}{5} + \frac{44}{5} = 56/5$

b) In the direction of the negative of the gradient.

$\nabla f(1,2) = \langle 4, 11 \rangle$

$|\nabla f(1,2)| = \sqrt{16+121} = \sqrt{137}$

direction:

$-\dfrac{\nabla f(1,2)}{|\nabla f(1,2)|} = \left\langle \dfrac{-4}{\sqrt{137}}, \dfrac{-11}{\sqrt{137}} \right\rangle$

c) Find $\vec{u}$ such that $D_{\vec{u}} f = 0$

i.e. $\vec{u}$ such that $\vec{u} \cdot \nabla f(1,2) = 0$.

two possibilities: (of unit vectors) $\left\langle \dfrac{11}{\sqrt{137}}, \dfrac{-4}{\sqrt{137}} \right\rangle, \left\langle \dfrac{-11}{\sqrt{137}}, \dfrac{4}{\sqrt{137}} \right\rangle$

4) Tangent plane to ellipsoid $x^2+4y^2 = 169 - 9z^2$ at $(3,2,4)$.

This is $f(x,y,z) = 169$ where $f = x^2+4y^2+9z^2$.

$f_x = 2x \qquad f_x(3,2,4) = 6$

$f_y = 8y \qquad f_y(3,2,4) = 16$

$f_z = 18z \qquad f_z(3,2,4) = 72$

Tangent plane: $f_x(3,2,4)(x-3) + f_y(3,2,4)(y-2) + f_z(3,2,4)(z-4) = 0$

$\rightarrow 6(x-3) + 16(y-2) + 72(z-4) = 0$

$\rightarrow 3(x-3) + 8(y-2) + 36(z-4) = 0$

$\rightarrow 3x + 8y + 36z = 169$

3) No.

$|D_{\vec{u}} f(1,2)| = |\nabla f(1,2) \cdot \vec{u}| = |\nabla f(1,2)||\vec{u}|\cos\theta = 5\cos\theta \leq 5.$

5) Find (x,y,z) such that tangent plane of

$$x^2+2y^2+4z^2+xy+3yz=1 \quad \text{is parallel to } xz\text{-plane.}$$

$$\nabla f = \langle 2x+y,\ 4y+x+3z,\ 8z+3y \rangle$$

Tangent plane is $\perp$ to ∇f.

So tangent plane parallel to xz-plane if & only if ∇f has x and z components zero.

Get two equations: $2x+y=0$, $\quad 8z+3y=0$,

$$\rightarrow y=-2x, \quad z=-\frac{3}{8}y=\frac{3}{4}x$$

Plug into ellipsoid equation:

$$x^2+2(-2x)^2+4\left(\frac{3}{4}x\right)^2+x(-2x)+3(-2x)\left(\frac{3}{4}x\right)=1$$

$$\rightarrow x^2+8x^2+\frac{9}{4}x^2-2x^2-\frac{9}{2}x^2=1 \rightarrow \frac{19}{4}x^2=1$$

$$\rightarrow x = \pm 2/\sqrt{19}$$

We obtain: $\left(2/\sqrt{19},\ -4/\sqrt{19},\ 3/2\sqrt{19}\right), \left(-2/\sqrt{19},\ 4/\sqrt{19},\ -3/2\sqrt{19}\right)$

6) $f(x,y) = x^2+y^2/2+x^2y$

$\nabla f = \langle 2x+2xy,\ y+x^2 \rangle$ Critical pts: $\nabla f = \vec{0}$

$$\rightarrow \begin{cases} 2x(1+y)=0 \\ y+x^2=0 \end{cases}$$

1st eq: $x=0$ or $y=-1$

$\downarrow$ 2nd eq $\quad \downarrow$ 2nd eq

$y=0 \quad x=\pm 1$

So critical points: $(0,0)$, $(1,-1)$, $(-1,-1)$

$f_{xx} = 2+2y \qquad f_{xy} = 2x \qquad f_{yy} = 1$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2 = 2+2y-4x^2$$

$(0,0)$: $\quad D(0,0) = 2 > 0. \quad f_{xx}(0,0) = 2 > 0.$ local min

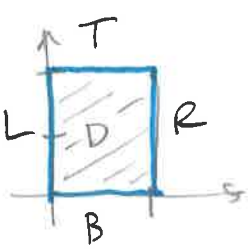
$(1,-1)$: $\quad D(1,-1) = -4 < 0,$ saddle

$(-1,-1)$: $\quad D(-1,-1) = -4 < 0,$ saddle

7) $f(x,y) = (x-1)^2 + (y-1)^2$ with domain $D = \{(x,y): 0 \le x \le 1, 0 \le y \le 2\}.$

$\nabla f = \langle 2(x-1), 2(y-1) \rangle \rightarrow$ Critical point $(1,1)$ is in D.

on the boundary: top side (T): $0 \le x \le 1, \; y = 2$



$f = (x-1)^2 + 1$, minimum at $x = 1$.
max@ $x = 0$

bottom side (B): $0 \le x \le 1, \; y = 0$

$f = (x-1)^2 + 1$, min @ $x = 1$.
max@ $x = 0$

left side (L): $x = 0, \; 0 \le y \le 2$

$f = 1 + (y-1)^2$, min @ $y = 1$
max@ $y = 0,2$

right side (R): $x = 1, \; 0 \le y \le 2$

$f = (y-1)^2$, min @ $y = 1$
max@ $y = 0,2$

side	min	max
T	1	2
B	1	2
L	1	2
R	0	1

@ critical pt,
$f(1,1) = 0.$

Minimum: 0 $\qquad$ Maximum: 2

8) maximize $f(x,y) = xy$ over $(x+1)^2 + y^2 = 1$

Use Lagrange mult. method. $\nabla f = \langle y, x \rangle$

$g(x,y) = (x+1)^2 + y^2 = 1$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases}$$

$\nabla g = \langle 2x+2, 2y \rangle$

$$\rightarrow \begin{cases} y = \lambda(2x+2) \\ x = \lambda(2y) \\ (x+1)^2 + y^2 = 1 \end{cases}$$

if $y \neq 0$, 2^{nd} eq: $\lambda = x/2y$

plug this into 1^{st} eq:

$y = \frac{x}{2y}(2x+2) \rightarrow 2y^2 = 2x^2 + 2x$

$\rightarrow y^2 = x(x+1)$

Then 3^{rd} eq gives:

$(x+1)^2 + x(x+1) = 1 \rightarrow x^2 + 2x + \cancel{1} + x^2 + x - \cancel{1} = 0$

$\rightarrow 2x^2 + 3x = 0$

$\rightarrow x = 0$ or $x = -3/2$

If $x = 0$, $y = 0$. If $x = -3/2$, $y = \pm\sqrt{(-3/2)(-3/2 + 1)} = \pm\sqrt{3}/2$

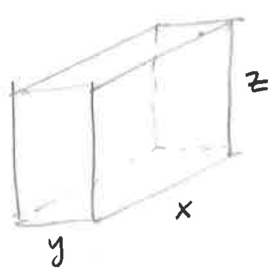
(If $y = 0$, note x must 0.)

So points found are $(0,0)$, $(-3/2, \pm\sqrt{3}/2)$

max

$f(0,0) = 0$, $f(-3/2, \sqrt{3}/2) = -3\sqrt{3}/4$, $f(-3/2, -\sqrt{3}/2) = 3\sqrt{3}/4$

9)



volume $= f(x,y,z) = xyz$

constraint: $g(x,y,z) = 4x + 4y + 4z = C$ (const.)

$\nabla f = \langle yz, xz, xy \rangle$

$\nabla g = \langle 4, 4, 4 \rangle$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 12 \end{cases} \rightarrow \begin{cases} yz = 4\lambda \\ xz = 4\lambda \\ xy = 4\lambda \\ 4x + 4y + 4z = C \end{cases}$$

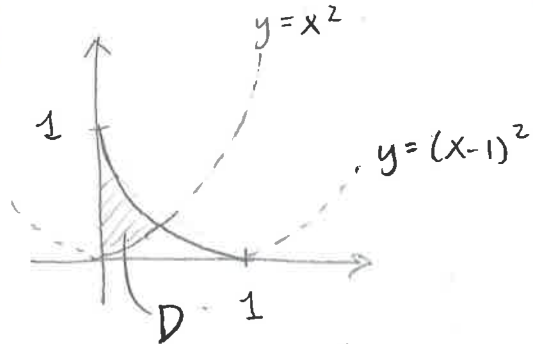
If any of x, y or z is 0, then volume is 0. But we're maximizing volume. So $x, y, z \neq 0$.

Then 1st and 2nd eq give $x = y$.

Similary 2nd, 3rd give $y = z$. So $x = y = z$.

4th eq. then gives $x = y = z = C/12$

10)



intersection point: $x^2 = (x-1)^2$

$\rightarrow x^2 = x^2 - 2x + 1$

$\rightarrow x = 1/2$

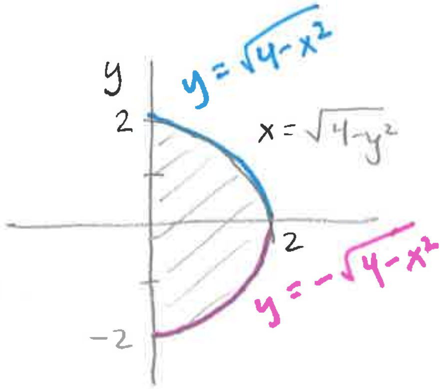
$$\iint_D (3x+1)\,dA = \int_0^{1/2} \int_{x^2}^{(x-1)^2} (3x+1)\,dy\,dx = \int_0^{1/2} ((x-1)^2 - x^2)(3x+1)\,dx$$

$$= \int_0^{1/2} (-2x+1)(3x+1)\,dx = \int_0^{1/2} (-6x^2 + x + 1)\,dx$$

$$= \left(-2x^3 + \frac{x^2}{2} + x\right)\Big|_0^{1/2} = -\frac{1}{4} + \frac{1}{8} + \frac{1}{2} = 3/8$$

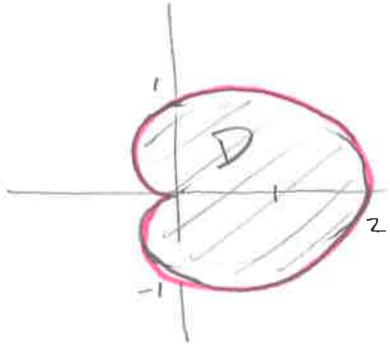
11)

$$\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy$$



$$= \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx$$

12) a) area of cardiod enclosed by $r = 1+\cos\theta$



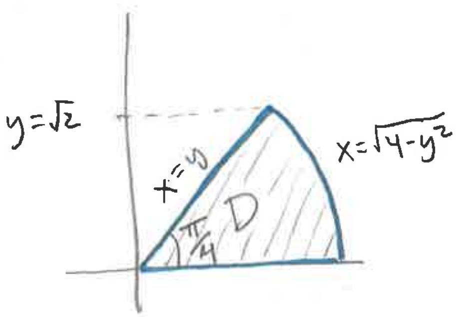
$$\iint_D dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{(1+\cos\theta)^2}{2}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1+2\cos\theta+\cos^2\theta)\,d\theta = \frac{1}{2}\int_0^{2\pi}\left(1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\Big|_{\theta=0}^{2\pi} = 3\pi/2$$

b)



$$\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} \frac{dx\,dy}{1+x^2+y^2} = \iint_D \frac{dA}{1+x^2+y^2}$$

$$= \int_0^{\pi/4}\int_0^{2} \frac{r\,dr\,d\theta}{1+r^2} \underset{\left(\substack{u=1+r^2 \\ du=2r\,dr}\right)}{=} \frac{1}{2}\int_0^{\pi/4}\int_1^{5} \frac{du\,d\theta}{u}$$

$$= \frac{1}{2}\int_0^{\pi/4} (\ln(5) - \cancel{\ln(1)})\,d\theta = \frac{\pi}{8}\ln(5)$$