Midterm 2 Solns MTH 211 Nov 2023

#1) a) $\boxed{x_t = \cos(t), \quad y_t = s}$

b) $f_t = f_x x_t + f_y y_t = \boxed{\cos(t) f_x + s f_y}$

c) $f_{tt} = \frac{\partial}{\partial t}\left(\cos(t) f_x + s f_y\right) = -\sin(t) f_x + \cos(t)\frac{\partial f_x}{\partial t} + s\,\frac{\partial f_y}{\partial t}$

$= -\sin(t) f_x + \cos(t)\left(f_{xx} x_t + f_{xy} y_t\right) + s\left(f_{xy} x_t + f_{yy} y_t\right)$

$= \boxed{-\sin(t) f_x + \cos^2(t) f_{xx} + 2s\cos(t) f_{xy} + s^2 f_{yy}}$

#2) $f(x,y) = 2xy + y^2 + 1$

$g(x,y) = x^2 + xy + \frac{1}{2}y^2 = 1$

$\nabla f = \langle 2y, 2x+2y \rangle \qquad \nabla g = \langle 2x+y, x+y \rangle$

$\begin{cases} \nabla f = \lambda \nabla g \\ g = 1 \end{cases} \rightarrow \begin{cases} 2y = \lambda(2x+y) \\ 2x+2y = \lambda(x+y) \\ g=1 \end{cases} \xrightarrow{2^{nd}\text{ eq}} (2-\lambda)(x+y)$

so either $x=-y$ or $\lambda = 2$.

If $x=-y$, $g=1$ becomes

$x^2 + x(-x) + \frac{1}{2}(-x)^2 = 1$

$\rightarrow x^2 = 2 \rightarrow x = \pm\sqrt{2}$, so get $(\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$.

If $\lambda = 2$, 1^{st} eq. gives $2y = 2(2x+y) \rightarrow x = 0$.

Then $g=1$ becomes $\frac{1}{2}y^2 = 1 \rightarrow y = \pm\sqrt{2}$. Get $(0,\sqrt{2}), (0,-\sqrt{2})$

$$\boxed{f(\sqrt{2},-\sqrt{2}) = f(-\sqrt{2},\sqrt{2}) = -1 \qquad f(0,\sqrt{2}) = f(0,-\sqrt{2}) = 3}$$

min max

#3) a) $\nabla f = \boxed{\langle ye^x + 1,\ e^x \rangle}$

b) $\vec{u} = \dfrac{\langle 1,-1\rangle}{|\langle 1,-1\rangle|} = \langle 1/\sqrt{2}, -1/\sqrt{2}\rangle$. $D_{\vec{u}} f(0,1) = \nabla f(0,1)\cdot \vec{u}$

$= \langle 2, 1\rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\rangle = \boxed{\frac{1}{\sqrt{2}}}$

c) $\dfrac{\nabla f(0,1)}{|\nabla f(0,1)|} = \dfrac{\langle 2,1\rangle}{\sqrt{4+1}} = \boxed{\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \rangle}$

d) $\boxed{\langle 1/\sqrt{5}, -2/\sqrt{5}\rangle,\ \langle -1/\sqrt{5}, 2/\sqrt{5}\rangle}$

#4) $\nabla f = \langle 3x^2 + 2y,\ 2x - 2y\rangle$ set to $= \langle 0, 0\rangle$:

$$\begin{cases} 3x^2 + 2y = 0 \\ 2x - 2y = 0 \end{cases} \xrightarrow[2^{nd}\text{ eq}]{} x = y \xrightarrow{1^{st}\text{ eq}} 3x^2 + 2x = 0$$

$x(3x+2) = 0$

so $x = 0$ or $x = -2/3$

We get the critical points $(0,0)$, $(-2/3, -2/3)$.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & 2 \\ 2 & -2 \end{vmatrix} = -12x - 4$$

$D(0,0) = -4 < 0$: $\boxed{(0,0) \text{ is a saddle}}$

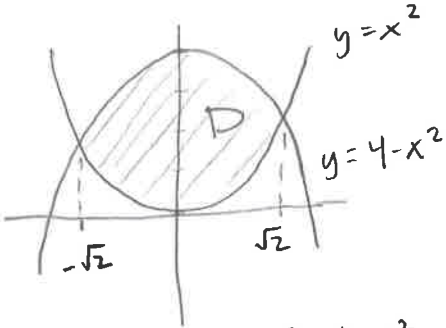
$D(-2/3, -2/3) = -12(-2/3) - 4 = 4 > 0$. $f_{xx}(-2/3, -2/3) = -4 < 0$,

so $\boxed{(-2/3, -2/3) \text{ is a local max}}$.

#5) $$\int_0^2\int_0^1 (2x + 3xy)\,dx\,dy = \int_0^2 \left(x^2 + \tfrac{3}{2}x^2 y\right)\Big|_{x=0}^{1} dy = \int_0^2 \left(1 + \tfrac{3}{2}y\right) dy$$

$$= y + \tfrac{3}{4}y^2 \Big|_{y=0}^{2} = 2 + \tfrac{3}{4}(2^2) = 2 + 3 = \boxed{5}$$

#6)



$x^2 = 4 - x^2$

$\rightarrow 2x^2 = 4$

$\rightarrow x = \pm\sqrt{2}$

$$\iint_D \frac{3x^2}{4-2x^2}\,dA = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{x^2}^{4-x^2} \frac{3x^2}{4-2x^2}\,dy\,dx = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{3x^2}{\cancel{4-2x^2}}\cancel{(4-x^2-x^2)}\,dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} 3x^2\,dx = x^3\Big|_{-\sqrt{2}}^{\sqrt{2}} = 2\sqrt{2} - (-2\sqrt{2}) = \boxed{4\sqrt{2}}$$

#7)

$$\text{Area} = 2\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{6\cos 2\theta}} r\,dr\,d\theta = \int_{-\pi/4}^{\pi/4} 6\cos 2\theta\,d\theta$$

$$= \frac{6}{2}\sin 2\theta\Big|_{\theta=-\pi/4}^{\pi/4} = 3(1-(-1)) = \boxed{6}$$