Midterm 1

MTH 211
Fall 2023

Name and ID number: ______________________________

No calculators, phones or any other devices may be present during the exam. Show work to receive full credit. There are 7 problems, equally weighted.

1. Let $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

   (a) Find a vector perpendicular to $\mathbf{v}$ and $\mathbf{w}$.

   (b) Find the area of a triangle spanned by $\mathbf{v}$ and $\mathbf{w}$.

(a) $$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = (-1-2)\vec{i} + (-(-1)+1)\vec{j} + (2-1)\vec{k}$$

$$= -3\vec{i} + 2\vec{j} + \vec{k}$$

(b) area of triangle $= \frac{1}{2}|\vec{v} \times \vec{w}| = \frac{1}{2}\sqrt{(-3)^2 + 2^2 + 1} = \frac{\sqrt{14}}{2}$

2. Consider the lines given by the following parametric curves:

$$\mathbf{r}_1(t) = \langle 1, 2, -1 \rangle + t\langle 0, -1, -1 \rangle, \qquad \mathbf{r}_2(s) = \langle 3, 1, -2 \rangle + s\langle 1, 1, 1 \rangle$$

(a) Find a point P where the two lines interesect. (Verify P is on both lines.)

(b) Find the angle between the two lines. (Your answer can have arccos.)

(a) $\vec{r}_1(t) = \langle 1, 2-t, -1-t \rangle \qquad \vec{r}_2(s) = \langle 3+s, 1+s, -2+s \rangle$

$$\begin{cases} 1 = 3+s & \longrightarrow \quad s = -2 \\ 2-t = 1+s & \longrightarrow \quad 2-t = 1-2 = -1 \rightarrow t = 3 \\ -1-t = -2+s & \end{cases}$$

Verify last eq is true w/ $s=-2$, $t=3$: $\quad -1-(3) = -2+(-2)$ ✓

Point of intersection is $\vec{r}_1(3) = \langle 1, -1, -4 \rangle$.

(b) direction vectors $\vec{a} = \langle 0, -1, -1 \rangle$, $\vec{b} = \langle 1, 1, 1 \rangle$

$\vec{a}\cdot\vec{b} = -2 \qquad |\vec{a}| = \sqrt{2} \qquad |\vec{b}| = \sqrt{3}$

$$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}\right) = \arccos\left(\frac{-2}{\sqrt{6}}\right)$$

3. Consider the function $f(x,y) = \frac{1}{2}x^2\tan(y)$.

(a) Find the tangent plane to the graph of $f(x,y)$ at $(x_0, y_0) = (1, \frac{\pi}{4})$.

(b) Approximate $f(1.1, \frac{\pi}{4} - 0.05)$ using the information from part (a).

(a) tangent plane: $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$(x_0,y_0) = (1, \frac{\pi}{4})$

$f(1,\frac{\pi}{4}) = \frac{1}{2}$ $\quad f_x = x\tan y \quad f_x(1,\frac{\pi}{4}) = 1$

$f_y = \frac{1}{2}x^2\sec^2 y \quad f_y(1,\frac{\pi}{4}) = \frac{1}{2}(1)^2\frac{1}{(\sqrt{2}/2)^2} = 1$

$\rightarrow$ tangent plane: $z = \frac{1}{2} + 1(x-1) + 1(y - \frac{\pi}{4})$

or $z = x + y - \frac{1}{2} - \frac{\pi}{4}$

(b) $f(1.1, \frac{\pi}{4} - 0.05) \approx \frac{1}{2} + 1(1.1-1) + 1(\cancel{\frac{\pi}{4}} - 0.05 - \cancel{\frac{\pi}{4}})$

$= 0.5 + 0.1 - 0.05 = 0.55$

4. The velocities of two moving particles in space are given by

$$\mathbf{r}_1'(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \mathbf{k}, \qquad \mathbf{r}_2'(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sqrt{t-1}\,\mathbf{k}$$

(a) Suppose $\mathbf{r}_1(0) = \mathbf{j} + \mathbf{k}$. Find the position function $\mathbf{r}_1(t)$.

(b) Find the arc length of $\mathbf{r}_2(t)$ from $t = 1$ to $t = 4$.

(a) $\vec{r}_1(t) = \langle \sin(t) + c_1,\ -\cos(t) + c_2,\ t + c_3 \rangle$

after integrating, where c_1, c_2, c_3 consts.

$\vec{r}_1(0) = \langle 0, 1, 1 \rangle$ gives $c_1 = 0,\ c_2 = 2,\ c_3 = 1$

thus $\vec{r}_1(t) = \langle \sin t,\ 2 - \cos t,\ t + 1 \rangle$

(b) $\ell = \int_1^4 |\vec{r}_2'(t)|\, dt = \int_1^4 \sqrt{\cos^2 t + \sin^2 t + (t-1)}\, dt$

$= \int_1^4 \sqrt{t}\, dt = \frac{2}{3} t^{3/2} \Big|_1^4 = \frac{2}{3}(4^{3/2} - 1^{3/2})$

$= \frac{2}{3}(8 - 1) = 14/3$

5. (a) Find the first partial derivatives of $f(x, y, z) = e^y \cos(x + zy)$.

(b) Evaluate

$$\frac{\partial^{100}}{\partial x^{97} \partial y^2 \partial x} \left(e^y + y^2 e^x + \cos(x)\right)$$

(a) $f_x = -e^y \sin(x + zy)$

$f_y = e^y \cos(x+zy) - e^y z \sin(x + zy)$

$f_z = -e^y y \sin(x + zy)$

(b) $$\frac{\partial^{100}}{\partial x^{98} \partial y^2} \left(e^y + y^2 e^x + \cos x\right)$$

$$= \frac{\partial^{99}}{\partial x^{98} \partial y} \left(e^y + 2y e^x\right)$$

$$= \frac{\partial^{98}}{\partial x^{98}} \left(e^y + 2e^x\right)$$

$= 2e^x$ since $\frac{\partial}{\partial x}(e^x) = e^x$, $\frac{\partial}{\partial x}(e^y) = 0$.

6. Determine whether the following four points in $\mathbb{R}^3$ lie in a common plane:

$$(1,0,1), \quad (2,-1,2), \quad (4,3,2), \quad (2,5,0)$$

P Q R S

$\vec{a} = \overrightarrow{PQ} = \langle 2-1, -1-0, 2-1 \rangle = \langle 1, -1, 1 \rangle$

$\vec{b} = \overrightarrow{PR} = \langle 4-1, 3-0, 2-1 \rangle = \langle 3, 3, 1 \rangle$

$\vec{c} = \overrightarrow{PS} = \langle 2-1, 5-0, 0-1 \rangle = \langle 1, 5, -1 \rangle$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} 1 & -1 & 1 \\ 3 & 3 & 1 \\ 1 & 5 & -1 \end{vmatrix} = -3 - 1 + 15 - 3 - 5 - 3 = 0$$

So they lie on a plane

7. Consider the line $\mathbf{r}(t) = \langle 1, 2, -1\rangle + t\langle 0, -1, -1\rangle$ and the plane given by

$$x + 2y - 3z = -1.$$

(a) Show the line and plane intersect in exactly one point (and find the point).

(b) find the angle between the line and the normal vector of the plane.

(a) $\vec{r}(t) = \langle 1, 2-t, -1-t \rangle$ $\quad x = 1, \; y = 2-t, \; z = -1-t$

plug into plane eq:

$$(1) + 2(2-t) - 3(-1-t) = -1$$

$$\rightarrow 1 + 4 - 2t + 3 + 3t = -1$$

$$\rightarrow t = -9$$

the point of intersection: $\vec{r}(-9) = \langle 1, 11, 8 \rangle$

(b) direction vector for line: $\vec{a} = \langle 0, -1, -1 \rangle$

normal vector for plane: $\vec{b} = \langle 1, 2, -3 \rangle$

$\vec{a} \cdot \vec{b} = -2 + 3 = 1$ $\quad |\vec{a}| = \sqrt{2}$ $\quad |\vec{b}| = \sqrt{1 + 2^2 + (-3)^2} = \sqrt{14}$

$$\theta = \arccos\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \arccos\left(\frac{1}{\sqrt{28}}\right)$$