Practice Problems – Final Exam

These problems do not represent all possible types of problems on the final exam. They focus on the topics covered in class after the second midterm. The exam will emphasize later material but may have earlier material as well. (Most content of the course is cumulative, after all.)

1. Factorize $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ into the form $A = X\Lambda X^{-1}$ where Λ is diagonal.

2. Determine whether each statement is true or false. If it is false, explain why.
   (a) The eigenvalues of an upper triangle matrix are the diagonal entries.
   (b) If an $n \times n$ matrix A has less than n distinct eigenvalues, then A cannot be diagonalizable.
   (c) Projection matrices are always invertible.
   (d) Orthogonal matrices, when acting as linear transformations, preserve lengths of vectors.

3. Let $x \in \mathbb{R}$. Consider the following matrix which depends on x:
$$A = \begin{bmatrix} 2x & -1 & 1 \\ x^2 & 1 & -1 \\ 1 & -x & -1 \end{bmatrix}$$
   Determine all x for which the column space of A has dimension less than 3.

4. Let $\lambda \in \mathbb{R}$ be some fixed scalar constant, and A an $n \times n$ matrix. Show that the eigenvectors of A with eigenvalue λ form a subspace of $\mathbb{R}^n$.

5. Find the plane P in $\mathbb{R}^3$ which contains the point $(0, 0, 2)$ and also best approximates the three points $(1, -1, 0)$, $(0, 1, 1)$, $(1, 0, 1)$. Here "best" is in the sense of least squares.

   To begin, note that a general plane can be written $ax + by + cz = d$ where $a, b, c, d \in \mathbb{R}$. Without much loss of generality we may assume $c = 1$. Requiring that P has the point $(0, 0, 2)$ determines d. There remain two unknowns: a, b. Plug in the three remaining points to obtain three equations in a, b. Apply least squares to this system of equations.[1]

6. Suppose a matrix A satisfies $A^T = A$. Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A with different eigenvalues: $A\mathbf{v_1} = \lambda_1\mathbf{v_1}$ and $A\mathbf{v_2} = \lambda_2\mathbf{v_2}$, with $\lambda_1 \neq \lambda_2$. Show $\mathbf{v_1}$ and $\mathbf{v_2}$ are orthogonal.[2]

[1] Least squares: the best solution to $A\mathbf{x} = \mathbf{b}$, when it cannot be solved, is given by solving instead $A^T A\mathbf{x} = A^T\mathbf{b}$.
[2] Useful: recall $\mathbf{v} \cdot \mathbf{w} = \mathbf{v}^T\mathbf{w}$. Then compute $A\mathbf{v}_1 \cdot \mathbf{v}_2$ in different ways, using identities you know.