Review for Final.

1. Factorize $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ into $X \Lambda X^{-1}$

where Λ = diagonal matrix.

Find eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{bmatrix}\right) = (1-\lambda)(3-\lambda)$$

$\lambda_1 = 1, \quad \lambda_2 = 3.$

eigenvectors:

$\underline{\lambda_1 = 1}$: solve $(A - \lambda_1 I)\vec{x} = \vec{0}$

$$A - \lambda_1 I = A - I = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}$$

one solution is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\underline{\lambda_2 = 3}$: solve $(A - \lambda_2 I)\vec{x} = \vec{0}$

$$A - \lambda_2 I = A - 3I = \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}$$

one solution is $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Then

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \det X = 1$$

$$X = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \qquad \det X = 1$$

$$X^{-1} = \frac{1}{\det X} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

So $A = X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$

2. T/F?

(a) T

(b) F ex. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diag but has only 1 e.v.

(c) F ex. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is projection to x-axis. it's not invertible.

(d) T

3. $x \in \mathbb{R}$

$$A = \begin{bmatrix} 2x & -1 & 1 \\ x^2 & 1 & -1 \\ 1 & -x & -1 \end{bmatrix}$$

For which $x \in \mathbb{R}$ is $\dim C(A) < 3$?

$\dim C(A) < 3 \iff$ columns of A are dependent $\iff$ $\det(A) = 0$.

$$\det A = -2x + \cancel{1} - x^3 - \cancel{1} - 2x^2 - x^2$$

$$= -x(2 + 3x + x^2) = -x(x+1)(x+2)$$

$\det A = 0$ exactly when $\boxed{x = 0, -1, -2}$.

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4. Eigenvectors of A w/ eigenvalue λ $= W$
(include $\vec{0}$)

$W = \{ \vec{v} : A\vec{v} = \lambda \vec{v} \}$

Need to show:

(i) if $\vec{u}, \vec{v}$ in W then $\vec{u} + \vec{v}$ is in W

(ii) if $\vec{v}$ is in W, $c \in \mathbb{R}$, then $c\vec{v}$ is in W.

(i): $\vec{u}, \vec{v}$ in W means $A\vec{u} = \lambda \vec{u}$, $A\vec{v} = \lambda \vec{v}$.

Then $A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = \lambda \vec{u} + \lambda \vec{v} = \lambda(\vec{u} + \vec{v})$ ✓

(ii): $\vec{v}$ in W means $A\vec{v} = \lambda \vec{v}$

Then $A(c\vec{v}) = cA\vec{v} = c\lambda\vec{v} = \lambda(c\vec{v})$. ✓

5. $P = \{ (x, y, z) : ax + by + cz = d \}$

Assume $c = 1$.

The point $(0, 0, 2)$ is on P: $\cancel{a0} + \cancel{b0} + 1(2) = d$

$\rightarrow d = 2$.

So the eq. for P is $ax + by + z = 2$?

$$
\begin{aligned}
(1,-1,0)&: \quad a - b = 2 \\
(0\ 1\ 1)&: \quad b + 1 = 2 \\
(1\ 0\ 1)&: \quad a + 1 = 2
\end{aligned}
\quad \longrightarrow \quad (*) \begin{cases} a - b = 2 \\ 0a + 1b = 1 \\ 1a + 0b = 1 \end{cases}
$$

$(*)$ is equiv. to $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Least Squares: solve $\boxed{A^T A \vec{x} = A^T \vec{b}.}$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A^TA} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ -1 \end{bmatrix}}_{A^T\vec{b}} \qquad \begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{(A^TA)^{-1}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$(A^TA)^{-1}$

$$= \frac{1}{3}\begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$\rightarrow a = 5/3,\ b = 1/3.$

So the plane P is $\boxed{\frac{5}{3}x + \frac{1}{3}y + z = 2}$.

6. Suppose $\boxed{A^T = A}$.

$\vec{v}_1, \vec{v}_2$ eigenvectors of A: $A\vec{v}_1 = \lambda_1\vec{v}_1$, $A\vec{v}_2 = \lambda_2\vec{v}_2$

(recall $\vec{v}_1 \cdot \vec{v}_2 = \vec{v}_1^T\vec{v}_2$) $\lambda_1 \neq \lambda_2$

Want to show: $\vec{v}_1 \perp \vec{v}_2 \iff \vec{v}_1 \cdot \vec{v}_2 = 0.$

$$(A\vec{v}_1) \cdot \vec{v}_2 = (\lambda_1\vec{v}_1) \cdot \vec{v}_2 = \lambda_1(\vec{v}_1 \cdot \vec{v}_2)$$

$$\|$$

$$(A\vec{v}_1)^T\vec{v}_2 = (\vec{v}_1^T A^T)\vec{v}_2 = \vec{v}_1^T(A^T\vec{v}_2) \overset{A^T = A}{=} \vec{v}_1^T(A\vec{v}_2)$$

$$= \vec{v}_1 \cdot (A\vec{v}_2)$$

$$= \vec{v}_1 \cdot (\lambda_2\vec{v}_2)$$

$$= \lambda_2(\vec{v}_1 \cdot v_2)$$

$\longrightarrow \lambda_1(\vec{v}_1 \cdot \vec{v}_2) = \lambda_2(\vec{v}_1 \cdot \vec{v}_2)$

$\longrightarrow (\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0 \longrightarrow \underline{\vec{v}_1 \cdot \vec{v}_2 = 0}.$ ✓

$$\longrightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0 \text{ since } \lambda_1 \neq \lambda_2} (\vec{v}_1 \cdot \vec{v}_2) = 0 \longrightarrow \vec{v}_1 \cdot \vec{v}_2 = 0.$$ ✓