Practice Problems – Midterm 2

1. Which of the following subsets of $\mathbb{R}^2$ are actually subspaces? Briefly explain each answer.
   (a) The vectors (a_1, a_2) with $a_1 a_2 = 0$.
   (b) The vectors with $a_1 + 3a_2 = 0$.
   (c) The vectors with $a_1^2 + a_2 = 0$.
   (d) The vectors with $a_1 < a_2$.
   (e) The vectors with $a_1 + a_2 = 1$.
   (f) All linear combinations of $(1, 2)$ and $(-1, 0)$.
   (g) The vectors with a_1, a_2 both integers.

2. Consider the vector space of 3×3 matrices. Write a typical matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

   Which of the following are subspaces?
   (a) Matrices with $a_{11} + a_{22} + a_{33} = 0$.
   (b) Matrices A such that $A\mathbf{x} = \mathbf{0}$ has only one solution.
   (c) Matrices A such that $A = A^T$, i.e. $a_{ij} = a_{ji}$ for all i, j.

3. In each case determine whether the given vectors are dependent or independent. Compute a basis for the subspace spanned by the vectors, and compute the dimension of this subspace.
   (a) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

4. Determine whether each statement is true or false.
   (a) For any matrix A, the column space $C(A)$ is equal to the row space $R(A)$.
   (b) The columns of any matrix A form a basis for $C(A)$, the column space of A.
   (c) For any matrix A, the dimension of $C(A)$ is equal to the dimension of $R(A)$.

(d) If A is an invertible $n \times n$ matrix, then $C(A) = \mathbb{R}^n$.

5. Use the rank nullity theorem to solve each problem.

   (a) Suppose the space of solutions to $A\mathbf{x} = \mathbf{0}$ is a plane in $\mathbb{R}^3$. What dimension is the column space of A?

   (b) Suppose a 110×54 matrix A has a column space with dimension 33. Compute the dimension of the space of solutions to $A\mathbf{x} = \mathbf{0}$.

6. For this problem recall the meaning of the sum of two subspaces; and also the formula that relates the dimensions of two subspaces, their intersection, and their sum.

   (a) Show that $U + W$ is a subspace.

   (b) Suppose there are two planes U, W in $\mathbb{R}^4$ that each pass through the origin. Suppose further that $U + W = \mathbb{R}^4$. What is $U \cap W$, and why?

   (c) Suppose that two subspaces $U, W \subset \mathbb{R}^n$ each of dimension 3 intersect in a line. What are the possibilities for n?

7. Let V_n be the vector space of polynomials of degree at most n. Let $T : V_n \to V_{n-1}$ be:

$$T(p(x)) = \frac{d}{dx}p(x)$$

   That is, T takes as input a polynomial $p(x)$ of degree at most n, and outputs its derivative, which is a polynonimal of degree at most $n - 1$.

   (a) Show T is a linear transformation.

   (b) Compute the nullspace of T.

   (c) Apply the Rank Nullity Theorem for T to compute the dimension of possible outputs.

   (d) Conclude: every polynomial is the derivative of another polynomial.