Practice Problems – Midterm 1

1. Use the cross product to find the solutions to the following system:

$$\begin{aligned} 2x + y - 3z &= 0 \\ x + 2y - 2z &= 0 \end{aligned}$$

Does the set of solutions intersect with the plane $x - y - z = 1$?

2. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Which of AB, BA, AC, CA, BC, CB make sense? Compute them.

3. Consider the equation $x + y - z = 1$. This is a plane in $\mathbb{R}^3$. Describe it in parametric form. Find the intersection of this plane with the line given by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} : \ t \in \mathbb{R} \right\}$$

4. Consider the following matrices:

$$A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

Compute A^2, A^3 and B^2, B^3. Make a prediction for A^n and B^n.

5. Consider vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^{15}$ satisfying $\mathbf{u} \cdot \mathbf{u} = 1$, $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{v} \cdot \mathbf{v} = 4$. Compute the angle between $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. (You can leave it in arccos form.)

6. Solve each of the following systems of linear equations by first doing elimination to get it into RREF. How many solutions are there? What is the geometric interpretation ("row picture")? How about the vector interpretation ("column picture")?

   (a)
$$\begin{aligned} x + y - z &= 0 \\ x - 2z &= 1 \\ x - y - z &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 - x_4 - x_5 &= 0 \\ x_1 + x_2 &= 1 \\ x_2 - x_3 - x_5 &= 0 \end{aligned}$$

(c)
$$\begin{aligned} x + 2y + 3z &= 1 \\ x - 2y + z &= -1 \\ x + 6y + 5z &= 3 \end{aligned}$$

(d)
$$\begin{aligned} x + 2y &= 0 \\ x + y &= 1 \\ 4x + 9y &= c \end{aligned}$$

Do (d) for a general constant c. Find for which c there are solutions, and for which c there are no solutions.

In each case, compute the expected dimension, and see if it agrees with the actual dimension of the solution space. (Recall: one solution (a point) is 0-dimensional, a line is 1-dimensional, a plane is 2-dimensional ... and something of "negative dimension" is an empty set.)

7. Consider the system of equations:

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 2 \\ & & y & + & z & = & 1 \\ x & - & y & - & 2z & = & -1 \end{array}$$

First write this system as a matrix-vector equation

$$A\mathbf{x} = \mathbf{b}.$$

Next, find A^{-1} using the algorithm we learned in lecture. Finally, use your answer for A^{-1} to find the solution to the original system of equations. (Do not directly do elimination on the original system of equations!)