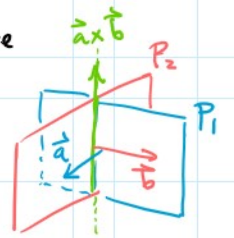
Review Problems

(1) Use $\times$ product to solve

$$\begin{cases} 2x + y - 3z = 0 & P_1 \\ x + 2y - 2z = 0 & P_2 \end{cases}$$



$\vec{b} \perp P_2$, $\vec{a} \perp P_1$

$$\vec{a} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Solution set is $\{ t(\vec{a}\times\vec{b}) : t \in \mathbb{R} \}$

$$\vec{a}\times\vec{b} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} (1)(-2) - (2)(-3) \\ -2(-2) + (-3)(1) \\ 2(2) - 1(1) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}$$

Solutions: $\left\{ t \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} : t \in \mathbb{R} \right\}$, a line.

Intersect with $x - y - z = 1$?

Plug in $x = 4t$, $y = t$, $z = 3t$:

$$(\cancel{4t}) - (\cancel{t}) - (\cancel{3t}) = 1$$

$$0 = 1$$

$\rightarrow$ no solutions, i.e. there is no intersection.

(2)

$$\begin{array}{ccc} 3\times 3 & 3\times 2 & 2\times 3 \\ A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix} & B = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ -1 & 1 \end{bmatrix} & C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \end{array}$$

Which of $\underset{3\times 2}{AB}$, $\cancel{BA}$, $\cancel{AC}$, $\underset{2\times 3}{CA}$, $\underset{3\times 3}{BC}$, $\underset{2\times 2}{CB}$ make sense?

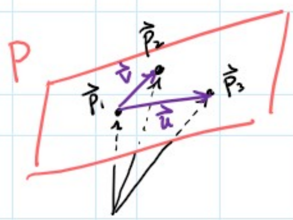
$$AB = \begin{bmatrix} 3 & 0 \\ 1 & 2 \\ 3 & 3 \end{bmatrix} \qquad CA = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 2 & 3 \end{bmatrix}$$

$$BC = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 0 & 3 \\ -1 & 2 & -2 \end{bmatrix} \qquad CB = \begin{bmatrix} 1 & 2 \\ 7 & -1 \end{bmatrix}$$

(3) Plane $x + y - z = 1$

Write in parametric form.

Some points on P:



$$\vec{p}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{p}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{p}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Let $\vec{p} = \vec{p}_1$, and $\vec{u} = \vec{p}_3 - \vec{p}_1 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$

$$\vec{v} = \vec{p}_2 - \vec{p}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Parametric form: $\{ \vec{p} + s\vec{u} + t\vec{v} : s, t \in \mathbb{R} \}$.

Another way: set $y = s$, $z = t$

$x + y - z = 1$

$x = 1 - y + z \longrightarrow x = 1 - s + t$

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - s + t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + s\underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

Find intersection of the plane with the line:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} : t \in \mathbb{R} \right\}$$

One way: plug $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 + t \\ -1 + 3t \end{bmatrix}$ into $x + y - z = 1$:

$$(1) + (2+t) - (-1+3t) = 1$$

$$4 - 2t = 1$$

$$-2t = -3 \rightsquigarrow t = 3/2 .$$

So $\vec{x} = \begin{bmatrix} 1 \\ 2+t \\ -1+3t \end{bmatrix} \underset{t=3/2}{=\!=} \begin{bmatrix} 1 \\ 2+3/2 \\ -1+9/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \\ 7/2 \end{bmatrix}$ is the intersection point.

(4) $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b+b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 A = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b+2b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3b \\ 0 & 1 \end{bmatrix}$$

$$A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 0 & 0 \end{bmatrix}$$

$$B^3 = B^2 B = \begin{bmatrix} 9 & 9 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 27 \\ 0 & 0 \end{bmatrix}$$

$$B^n = \begin{bmatrix} 3^n & 3^n \\ 0 & 0 \end{bmatrix}$$

(6) (a) $\begin{cases} x + y - z = 0 \\ x - 2z = 1 \\ x - y - z = 0 \end{cases}$ expected dim =

$$\begin{cases} x - 2z = 1 \\ x - y - z = 0 \end{cases}$$

expected dim = #unknowns − # eqs
= 3 − 3 = 0.

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 0 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{-1} & -1 & 1 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{-1} & -1 & 1 \\ 0 & 0 & \boxed{2} & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 0 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \text{ RREF}$$

Solutions:

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$$

Geometry: 3 planes in $\mathbb{R}^3$ intersect in one point.

(expected dimension (=0) agrees with actual dim.)

Col. Picture: Original system is equivalent to:

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We found this
has a unique solution: $x = -1$, $y = 0$, $z = -1$.

(d)
$$\begin{cases} x + 2y = 0 \\ x + y = 1 \\ 4x + 9y = c \end{cases}$$

c some constant.

expected dim = # unknowns − # eqs
= 2 − 3 = −1

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 1 & 1 & 1 \\ 4 & 9 & c \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & \boxed{-1} & 1 \\ 0 & 1 & c \end{array}\right] \to \left[\begin{array}{cc|c} \boxed{1} & 0 & -2c \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 1+c \end{array}\right]$$ RREF

If $c \neq -1$ then there are no solutions. (expected dim. -1 agrees w/ actual dim) "dim(no sols) $= -1$"

If $c = -1$:

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x = 2 \\ y = -1 \end{array} \to \text{one solution } \vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(exp. dim -1 doesn't agree with actual, 0) dimension of a point.

Geometry: 3 lines in $\mathbb{R}^2$
when $c \neq -1$ there is no common intersection
when $c = -1$ there is a single intersection

col. picture:

$$x\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$$

no soln. when $c \neq -1$; when $c = -1$ there is the one solution $x = 2$, $y = -1$.

(b)

$$\begin{cases} x_1 + 2x_2 - x_4 - x_5 = 0 \\ x_1 + x_2 = 1 \\ x_2 - x_3 - x_5 = 0 \end{cases}$$

exp. dim $=$ # unk $-$ # eqs $= 5 - 3 = 2$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -1 & -1 & 0 \\ 0 & \boxed{-1} & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -1 & -1 & 0 \\ 0 & \boxed{-1} & 0 & 1 & 1 & 1 \\ 0 & 0 & \boxed{-1} & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccccc|c} \boxed{1} & 2 & 0 & -1 & -1 & 0 \\ 0 & \boxed{1} & 0 & -1 & -1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 0 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 0 & 0 & 1 & 1 & 2 \\ 0 & \boxed{1} & 0 & -1 & -1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 0 & -1 \end{array}\right]$$ RREF

Free variables: x_4, x_5 Let $x_4 = s$, $x_5 = t$
Pivot variables: x_1, x_2, x_3

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Then
$$\begin{aligned} x_1 + s + t &= 2 \\ x_2 - s - t &= -1 \\ x_3 - s &= -1 \end{aligned}$$

$$\rightsquigarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2-s-t \\ -1+s+t \\ -1+s \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + s \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

The solutions form a plane. (exp. dim 2 = actual dim.)

Geometry: 3 hyperplanes in $\mathbb{R}^5$ intersect to give a 2-dimensional plane.

col. picture: Original system can be written

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and we found the solutions as described above.

(c)
$$\begin{cases} x + 2y + 3z = 1 \\ x - 2y + z = -1 \\ x + 6y + 5z = 3 \end{cases}$$

exp. dim. = # unk - # eqs = 3 - 3 = 0.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 1 & -2 & 1 & -1 \\ 1 & 6 & 5 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-4} & -2 & -2 \\ 0 & 4 & 2 & 2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-4} & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{1} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

Pivot vars.: x, y

Free vars.: z

Let $z = t$

$$\begin{aligned} x + 2t &= 0 \\ y + \tfrac{1}{2}t &= \tfrac{1}{2} \end{aligned} \quad \rightsquigarrow \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

$$x + 2t = 0$$
$$y + \tfrac{1}{2}t = \tfrac{1}{2}$$

$\rightsquigarrow$ $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Solutions form a line.

Note exp. dim $= 0$ is not actual dim. $= 1$.

Geometry: 3 planes in $\mathbb{R}^3$ intersect in a line

Col. picture: The vector equation

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -2 \\ 6 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

has the solutions described above.

(5) Find angle θ between $\vec{u}, \vec{v}$ if given:
$\vec{u} \cdot \vec{u} = 1$, $\vec{u} \cdot \vec{v} = 0$, $\vec{v} \cdot \vec{v} = 4$.

Compute $(\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v}) = \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$
$= 1 + 0 + 4$
$= 5$

$(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} - 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}$
$= 1 - 0 + 4$
$= 5$

$(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v}) = \vec{u} \cdot \vec{u} - \vec{v} \cdot \vec{v}$
$= 1 - 4 = -3$

So: $\cos\theta = \dfrac{(\vec{u}+\vec{v}) \cdot (\vec{u}-\vec{v})}{\|\vec{u}+\vec{v}\| \, \|\vec{u}-\vec{v}\|} = \dfrac{-3}{\sqrt{5}\sqrt{5}} = \dfrac{-3}{5}.$

$$\theta = \arccos\left(\frac{-3}{5}\right)$$

(7) Consider the system

$$(*) \begin{cases} x + y - z = 2 \\ y + z = 1 \\ x - y - 2z = -1 \end{cases}$$

Write this as $A\vec{x} = \vec{b}$. Compute A^{-1} and use this to solve the system.

Write this as $A\vec{x} = \vec{b}$. Compute A^{-1} and use this to solve the system.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

With these choices, (*) is $A\vec{x} = \vec{b}$. Let's compute A^{-1}:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & -1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -1 & 2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & 0 & 2 & 1 \\ 0 & \boxed{1} & 0 & 1 & -1 & -1 \\ 0 & 0 & \boxed{1} & -1 & 2 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{matrix}}_{I} & \underbrace{\begin{matrix} -1 & 3 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{matrix}}_{A^{-1}} \end{array}\right]$$

The solution to (*) is $A^{-1}\vec{b}$, which we compute:

$$A^{-1}\vec{b} = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$