Midterm 2 Solutions

1. Which of these subsets of $\mathbb{R}^3$ are subspaces?

(a) Vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that $|a_1| > 1$.

No. $\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is in the collection $(|a_1| = 2 > 1)$

but $\left(\frac{1}{2}\right)\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not (since $|1| = 1 \not> 1$)

Alternatively: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the collection!

(b) Vectors with $a_1 + a_2 - a_3 - 1 = 0$.

No. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the collection:

$$0 + 0 - 0 - 1 = -1 \neq 0.$$

(c) Vectors with $a_1 + a_2 + a_3 = 0$.

Yes. This is a plane through the origin.

(d) Vectors for which a_1, a_2, a_3 are rational.

No. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is in the collection of vectors

$\begin{bmatrix} \\ 0 \end{bmatrix}$

but the scaling $\pi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi \\ 0 \\ 0 \end{bmatrix}$ is not.

($a_1 = \pi$ is irrational.)

(e) Vectors with $a_1 = 2a_2$ and $a_1 = 3a_3$.

Yes. These vectors are $\left\{ t \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} \right)$
(let $t = a_1$)

which is a line that passes through $\vec{0}$.

2. Determine (in)dependendence, compute a basis for the span, and find the dimension of the spanned subspace.

(a) $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 6 \\ 15 \\ 21 \\ 0 \\ 3 \end{bmatrix}$

Note $\vec{v}_2 = 3\vec{v}_1$.
So they are dependent

$\operatorname{Span}(\vec{v}_1, \vec{v}_2)$ has basis $\vec{v}_1$ (or $\vec{v}_2$), and dimension 1.

(b) $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ \cdot & \cdot & 1 \end{bmatrix} \xrightarrow{\text{elim.}} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ & & \boxed{} \end{bmatrix}$ all pivot columns

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{\text{elim.}} \begin{bmatrix} \boxed{1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$ all pivot columns

→ they are independent, basis is $\vec{v}_1, \vec{v}_2, \vec{v}_3$ and dimension is 3.

(c) $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -4 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 4 \\ -8 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -1 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} -4 \\ 3 \\ -5 \\ -5 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 2 & -4 \\ 3 & 6 & 1 & 3 \\ 2 & 4 & 3 & -5 \\ -4 & -8 & -1 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & -4 \\ 0 & 0 & -5 & 15 \\ 0 & 0 & \boxed{-1} & 3 \\ 0 & 0 & 7 & -21 \end{bmatrix} \longrightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & -4 \\ 0 & 0 & \boxed{-1} & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns 1, 3

→ they're dependent
$\vec{v}_1, \vec{v}_3$ is a basis
dimension is .

(3) Find basis / dimension of each subspace of $\mathbb{R}^3$.

(a) vectors whose components are equal.

basis: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ dimension = 1.

(b) vectors whose components add to zero.

(b) vectors whose components add to zero.

basis: $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ dimension: 2

(c) Solve $A\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 2 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

z is free variable, say t.

$y = 0. \quad x + t = 0 \rightarrow x = -t.$

So solutions are $\left\{ \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = N(A)$

Basis: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ Dimension $= 1$

(d) $\text{span}\left(\underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}}_{\text{this is a basis}} \right)$ dimension $= 2$.

4. Rank Nullity Theorem for A $m \times n$ matrix:

$\dim C(A) + \dim N(A) = n$

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(a) A is $m \times n = 32 \times 27$ so $n = 27$

$\dim C(A) = 25$, so

$$\dim N(A) = n - \dim C(A) = 27 - 25 = 2.$$

(b) $\dim N(A) = 1$ and $n = 5$. So

$$\dim C(A) = n - \dim N(A) = 5 - 1 = 4.$$

(c) A is 7×7 so $n = 7$.

$$\dim C(A) + \dim N(A) = 7$$

One must be even, one must be odd.

so they can't be equal.

5. True or False: all are true.