Midterm 1 Solutions

(1) $\begin{cases} x+2y-z=0 \\ x+y-2z=1 \\ x-y-z=0 \end{cases} \longrightarrow \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 1 & 1 & -2 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right]$

subtract 1^{st} row from 2^{nd}, 3^{rd} rows.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{-1} & -1 & 1 \\ 0 & -3 & 0 & 0 \end{array}\right]$$

add $-3\times \text{row}_2$ to row_3.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{-1} & -1 & 1 \\ 0 & 0 & \boxed{3} & -3 \end{array}\right]$$

scale row_2 by -1.
row_3 by $\frac{1}{3}$.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 0 \\ 0 & \boxed{1} & 1 & -1 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

subtract row_3 from row_2.
add row_3 to row_1.

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

subtract $2\times \text{row}_2$ from row_1.

RREF $$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right]$$

Solution: there's one solution $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$
(geometrically: a point)

(a) Expected dimension = #unknowns − #eqs
$= 3 - 3 = 0$

This agrees with the dimension of the actual solution set, which is a point.

(b) Column picture:

$$x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + z\begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

has only the solution $x = -1,\ y = 0,\ z = -1$.

(2) Plane P in $\mathbb{R}^3$ given by $2x - y + z = 3$.

(a) Let $y = s$, $z = t$. Then $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on this plane is equal to

$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3 + s - t) \\ s \\ t \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix}$$

$$= \underset{\vec{p}}{\begin{bmatrix} 3/2 \\ 0 \\ 0 \end{bmatrix}} + s \underset{\vec{u}}{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}} + t \underset{\vec{v}}{\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}}$$

Thus $P = \{ \vec{p} + s\vec{u} + t\vec{v} : \ s, t \in \mathbb{R} \}$

where $\vec{p}$, $\vec{u}$, $\vec{v}$ are indicated above.

(b) By fixing s or t to be a constant we get a line in P.
For example, let $t = 0$. Then

$$\{ \vec{p} + s\vec{u} : s \in \mathbb{R} \} = \left\{ \begin{bmatrix} 3/2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} : s \in \mathbb{R} \right\}$$

is a line contained in the plane P.

(3) angle formula: $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$

Given: $\vec{u} \cdot \vec{u} = 2$, $\vec{u} \cdot \vec{v} = -1$, $\vec{v} \cdot \vec{v} = 1$.

Find the angle θ between $\vec{u}$ and $2\vec{u} - \vec{v}$.

$$\vec{u} \cdot (2\vec{u} - \vec{v}) = 2\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} = 2(2) - (-1)$$

$$\vec{u}\cdot(2\vec{u}-\vec{v}) = 2\vec{u}\cdot\vec{u} - \vec{u}\cdot\vec{v} = 2(2)-(-1)$$
$$= 5$$

$$(2\vec{u}-\vec{v})\cdot(2\vec{u}-\vec{v}) = 4\vec{u}\cdot\vec{u} - 4\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$
$$= 4(2) - 4(-1) + 1$$
$$= 13$$

$$\cos\theta = \frac{\vec{u}\cdot(2\vec{u}-v)}{\sqrt{\vec{u}\cdot\vec{u}}\,\sqrt{(2\vec{u}-\vec{v})\cdot(2\vec{u}-\vec{v})}} = \frac{5}{\sqrt{2}\sqrt{13}}$$

$$\theta = \arccos\left(\frac{5}{\sqrt{26}}\right).$$

(4)

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 1 \\ x_1 - x_2 + x_3 + x_4 = 0 \end{cases} \longrightarrow \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 0 \end{array}\right]$$

subtract row$_1$ from rows 2 & 3 $\downarrow$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{-2} & 0 & -2 & 1 \\ 0 & -2 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{subtract row}_2\text{ from row}_3} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{-2} & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 & -1 \end{array}\right]$$

$\downarrow$ $(-\frac{1}{2})\times \text{row}_2$, $(\frac{1}{2})\times\text{row}_3$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{2} \end{array}\right] \xrightarrow[\text{rows } 1 \,\&\, 2]{\text{subtract row}_3 \text{ from}} \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & \frac{1}{2} \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & \boxed{1} & -\frac{1}{2} \end{array}\right] \xrightarrow[\text{1 \& 2}]{\text{rows}} \left[\begin{array}{cccc|c} 0 & 0 & 0 & \boxed{1} & -\frac{1}{2} \end{array}\right]$$

$$\downarrow$$

RREF
$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & \frac{1}{2} \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & -\frac{1}{2} \end{array}\right]$$

Pivot variables: x_1, x_2

Free variables: $x_3 = t$

$$x_1 + x_3 = \frac{1}{2} \longrightarrow x_1 = \frac{1}{2} - t$$

$$x_2 = 0$$

$$x_4 = -\frac{1}{2}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - t \\ 0 \\ t \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \text{ any real \#}$$

Geometry: 3 hyperplanes in $\mathbb{R}^4$ intersect in a line.

(5)

$$\underset{3\times 3}{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 0 \\ 2 & 0 & 2 \end{bmatrix} \qquad \underset{3\times 2}{B} = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underset{2\times 2}{C} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Among AB, BA, BC, CB, AC, CA only $\underset{3\times 2}{AB}$, $\underset{3\times 2}{BC}$ make sense.

$$AB = \begin{bmatrix} 2 & 2 \\ 9 & 0 \\ 4 & 4 \end{bmatrix} \qquad BC = \begin{bmatrix} 2 & 2 \\ 0 & 3 \\ 2 & 0 \end{bmatrix}$$

Now let's compute C^2 and C^n (n = a positive integer).

$$C^2 = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I \quad (I: \text{identity})$$

Then for $n = 2m$ even, $C^n = C^{2m} = (C^2)^m$

$$= (2I)^m = 2^m I = \begin{bmatrix} 2^m & 0 \\ 0 & 2^m \end{bmatrix}.$$

If $n = 2m+1$ is odd then

$$C^n = C^{2m+1} = C^{2m}C = \begin{bmatrix} 2^m & 0 \\ 0 & 2^m \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2^m \\ 2^{m+1} & 0 \end{bmatrix}.$$