Review of Determinants

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdot & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

M_{ij} = $(n-1)\times(n-1)$ matrix obtained by deleting i^{th} row & j^{th} column of A.

$$\boxed{\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{ni}}$$

"cofactor formula"

formula works for any i

where $C_{ij} = (-1)^{i+j}\det(M_{ij})$.

(i,j) cofactor.

Ex.

$$\underset{4\times 4}{A} = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

M_{12} $\quad M_{13}$

$$\det A = \cancel{a_{11}}^{\,0}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cancel{a_{14}}^{\,0}C_{14}$$

$$= C_{12} - C_{13}$$

$$C_{12} = (-1)^{1+2}\det M_{12} = -\det\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -(1+0+0-0-0-(-1))$$

$$= -2$$

$$C_{13} = (-1)^{1+3}\det M_{13} = \det\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\rightarrow \det A = (-2) - (0) = -2$$

Cofactors also help compute inverses:

$\underset{n\times n}{A}$ invertible: $\boxed{\text{the } (i,j)\text{-entry of } A^{-1} \text{ is } C_{ji}/\det(A)}$

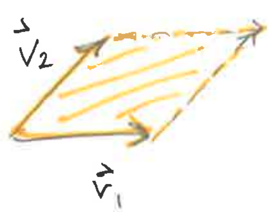
This gives an alternative way to compute A^{-1}

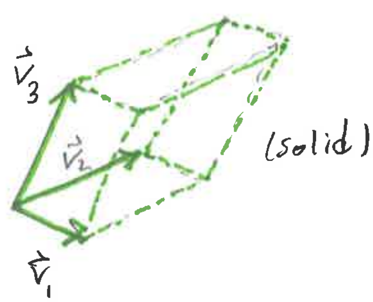
(We learned before how to use elimination to find A^{-1}.)

<u>Geometric Interpretation of det:</u>

$\vec{v}_1, \dots, \vec{v}_n$ vectors in $\mathbb{R}^n$

$$\underset{\text{parallelepiped}}{P} = \{ t_1\vec{v}_1 + \cdots + t_n\vec{v}_n \mid t_1, \dots, t_n \in [0,1] \}$$





then volume $(P) = \left| \det\left(\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \right) \right|$.

When $n=3$ we in fact have

$$\underbrace{(\vec{v}_1 \times \vec{v}_2)}_{\text{cross product}} \cdot \vec{v}_3 = \det\left(\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \right).$$

Eigenvalues & Eigenvectors

diagonal matrix $\underset{n\times n}{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$ Easy to understand geometrically!

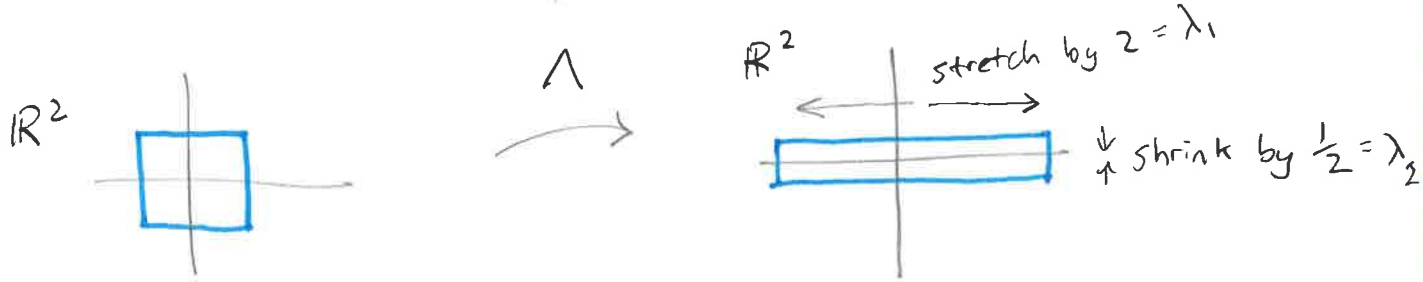
$\Lambda : \mathbb{R}^n \to \mathbb{R}^n$

linear transformation

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ in $\mathbb{R}^n \overset{\Lambda}{\rightsquigarrow} \Lambda\vec{x} = \begin{bmatrix} \lambda_1 x_1 \\ \lambda_2 x_2 \\ \vdots \\ \lambda_n x_n \end{bmatrix}$

Each coordinate is scaled separately.

Ex. $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

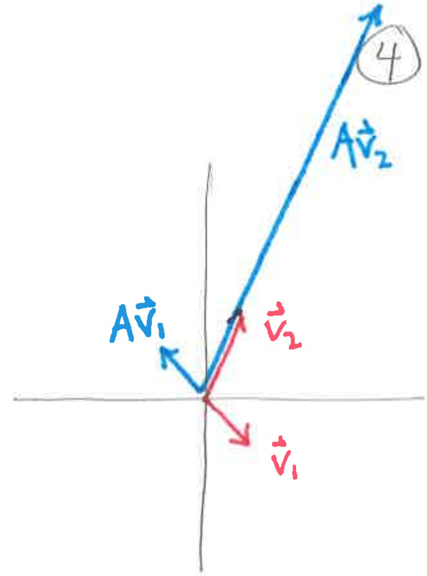


Given any $n\times n$ matrix A can we somehow "view it" as a diagonal matrix?

Ex. $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A\vec{v}_1 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\vec{v}_1$$

$$A\vec{v}_2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 5\vec{v}_2$$



Note $\vec{v}_1, \vec{v}_2$ is a basis of $\mathbb{R}^2$

A determined by where it sends $\vec{v}_1, \vec{v}_2$

$A: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{v}_1 \longrightarrow -\vec{v}_1$

$\vec{v}_2 \longrightarrow 5\vec{v}_2$

A is "diagonal from the viewpoint of $\vec{v}_1, \vec{v}_2$"

$\lambda_1 = -1$

$\lambda_2 = 5$

Break into 3 steps:

(1) Send each $\vec{v}_1, \vec{v}_2$ to standard basis vectors $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\vec{v}_1 \longrightarrow \vec{e}_1$

$\vec{v}_2 \longrightarrow \vec{e}_2$

call the matrix X^{-1}

so $X^{-1}\vec{v}_1 = \vec{e}_1, \quad X^{-1}\vec{v}_2 = \vec{e}_2$

(2) Scale each of $\vec{e}_1, \vec{e}_2$ by λ_1, λ_2

$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$e_1 \longrightarrow \lambda_1 e_1$

$e_2 \longrightarrow \lambda_2 e_2$

call the matrix Λ

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

In our example, $\lambda_1 = -1, \ \lambda_2 = 5.$

(3) send $\vec{e}_1, \vec{e}_2$ back to $\vec{v}_1, \vec{v}_2$

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \vec{e}_1 &\longrightarrow \vec{v}_1 \\ \vec{e}_2 &\longrightarrow \vec{v}_2 \end{aligned}$$

the matrix is X

in fact: $X = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$.

Then we have:

$$A\vec{x} = X \Lambda X^{-1} \vec{x} \quad \text{for all } \vec{x} \text{ in } \mathbb{R}^2$$

or in other words: $\boxed{A = X \Lambda X^{-1}}$

diagonal

This is the process of diagonalizing A.

change of basis matrices

Our example:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad X = \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \\ \multicolumn{2}{c}{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}} \end{array}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$X^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

Check $A = X \Lambda X^{-1}$:

$$X \Lambda X^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -1 & 5 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 6 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = A. \checkmark$$

General Case:

$\underset{n\times n}{A}$ If $A\vec{x}=\lambda\vec{x}$ for some $\vec{x}\neq\vec{0}$ then λ is an eigenvalue of A, and $\vec{x}$ is an eigenvector associated to λ.

λ eigenvalue of A $\iff$ $A\vec{x}=\lambda\vec{x}$ for some $\vec{x}\neq\vec{0}$ $\iff$ $(A-\lambda \underset{n\times n \text{ identity}}{I})\vec{x}=\vec{0}$ for some $\vec{x}\neq\vec{0}$

$\iff$ $A-\lambda I$ is not invertible $\iff$ $\det(A-\lambda I)=0$.

Thus:

$$\boxed{\text{Eigenvalues of } A \text{ are the roots of } \det(A-\lambda I).}$$

Here we think of $\det(A-\lambda I)$ as a polynomial in λ.

Ex: $A=\begin{bmatrix}1&2\\4&3\end{bmatrix}$ $\quad A-\lambda I=\begin{bmatrix}1&2\\4&3\end{bmatrix}-\begin{bmatrix}\lambda&0\\0&\lambda\end{bmatrix}=\begin{bmatrix}1-\lambda&2\\4&3-\lambda\end{bmatrix}$

$$\det(A-\lambda I)=\det\left(\begin{bmatrix}1-\lambda&2\\4&3-\lambda\end{bmatrix}\right)=(1-\lambda)(3-\lambda)-(2)(4)$$

$$=3-4\lambda+\lambda^2-8=\lambda^2-4\lambda-5=(\lambda-5)(\lambda+1)$$

The roots are $-1, 5$. Agrees with λ_1, λ_2 from before!

To find an eigenvector associated to eigenvalue λ:

$$\boxed{A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}}$$

Just solve the equation for $\vec{x}$!

Ex: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ eigenvalues?

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$

Eigenvector for $\lambda_1 = 1$:

$$A - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

solve $(A - \lambda_1 I)\vec{x} = \vec{0}$ i.e. $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution, hence an eigenvector associated to eigenvalue $\lambda_1 = 1$.

Eigenvector for $\lambda_2 = -1$:

$A - \lambda_2 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ an eigenvector is $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$