Determinants (cont.)

- $\det(\underset{n\times n}{A}\ \underset{n\times n}{B}) = \det(A)\det(B)$

- $\det\left(\underbrace{\begin{bmatrix} a_1 & ? & ? \\ 0 & a_2 & ? \\ \vdots & 0 & \ddots \\ 0 & \cdots & 0\ a_n \end{bmatrix}}_{\text{upper triangular}}\right) = a_1 a_2 \cdots a_n$

Similarly the det of lower triangular matrix is product of diagonal entries

(1)

$$\overset{A}{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}} = \underbrace{\overset{B}{\begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{2}{3} & 1 \end{bmatrix}}}_{\text{lower triang}} \underbrace{\overset{C}{\begin{bmatrix} 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}}_{\text{upper triang.}}$$

$\det(A) = \det(BC) = \det(B)\det(C) = (1)(4) = 4.$

$$\left(\begin{array}{l} \det(B) = (1)(1)(1) = 1 \\ \det(C) = (\cancel{2})(\tfrac{\cancel{3}}{\cancel{2}})(\tfrac{4}{\cancel{3}}) = 4 \end{array}\right)$$

(2)

$$\det\left(\begin{bmatrix} 3 & 2 & 500 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\right) = (3)(1)(2) = 6$$

$\det\left(\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}}_{2I}\right) = 4$

$c \in \mathbb{R}$

$\det(cA) \neq c\det(A)$ in general.

$$\det(cI) = \det\left(\overset{n\times n}{\begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & c \end{bmatrix}}\right) = (c)(c)\cdots(c) = c^n$$

In general $\det(\underset{c\in\mathbb{R}}{c}\underset{n\times n}{A}) = c^n \det(A)$

Recall: for any $n \times n$ matrix A, can factor

$$A = \underset{\substack{\text{permutation} \\ \text{matrix}}}{\overset{n\times n}{P}} \; \underset{\substack{\text{lower} \\ \text{triang.}}}{\overset{n\times n}{L}} \; \underset{\substack{\text{upper} \\ \text{triang.}}}{\overset{n\times n}{U}}$$

Thus: $\det(A) = \det(PLU)$

$$= \underset{???}{\det(P)} \; \underbrace{\det(L) \; \det(U)}_{\text{easy to compute!}}$$

A permutation matrix P is obtained by doing a sequence of row swaps.

Ex. $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{swap rows } 1 \,\&\, 3]{P_{13}} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[\text{swap rows } 2 \,\&\, 3]{P_{23}} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = P$$

$$P = P_{23} P_{13} = \overset{P_{23}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \overset{P_{13}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}$$

Fact: $\det(P_{ij}) = -1$.

row swap of rows i, j

So in our example: $\det(P) = \det(P_{23} P_{13})$

$$= \det(P_{23}) \det(P_{13})$$

$$= (-1)(-1) = 1$$

In general, permutation matrix

$$P = \underbrace{P_{i_1 i_2} P_{i_3 i_4} \cdots P_{i_{k-1} i_k}}_{m \,=\, \#\text{ row swaps of } P}$$

$$\det(P) = (-1)^m = \begin{cases} 1 & \text{if } m \text{ even} \\ -1 & \text{if } m \text{ odd} \end{cases}$$

Example of fact:

$$0 = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) + \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\right)$$

$$= \cancel{\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)} + \underbrace{\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)}_{=1} + \det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right) + \cancel{\det\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)}$$

$$\longrightarrow \det\left(\underset{P_{23}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}\right) = -1 \qquad \text{i.e.} \quad \det(P_{23}) = -1.$$

Ex.

$$\underset{A}{\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}} = \underset{P = P_{13}}{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}$$

$$\det(P) = -1 \qquad \det(L) = 1 \qquad \det(U) = (-1)(1)(-1) = 1$$

$$\det(A) = \det(P)\det(L)\det(U) = (-1)(1)(1) = -1.$$

"Big Formula": first by example:

$$\det\left(\begin{bmatrix} 2 & 1 & 4 \\ 2 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}\right) = (2)(3)(1) + (1)(-1)(1) + (4)(2)(1)$$

$$- (1)(3)(4) - (1)(-1)(2) - (1)(2)(1)$$

$$= 6 - 1 + 8 - 12 + \cancel{2} - \cancel{2} = 1$$

$$\det\left(\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}\right) = 0 + 0 + (-1)(1)(1) - 0 - 0 - 0 = -1$$

Note a permutation P just interchanges the coordinates:

$$P\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_\alpha \\ x_\beta \\ \\ x_\omega \end{bmatrix}$$

Big Formula:

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{1n} \\ a_{12} & & \ddots & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$$

$$\det(A) = \overset{\text{sum}}{\sum_{\substack{\text{all} \\ \text{permutations} \\ P\ (n\times n)}}} \overbrace{\det(P)}^{\pm 1}\, a_{1\alpha}\, a_{2\beta} \cdots a_{n\omega}$$

3x3 case:

$$\det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right) = \overset{\begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}}{\underline{a_{11}a_{22}a_{33}}} + \overset{\begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}}{\underline{a_{12}a_{23}a_{31}}} + \overset{\begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}}{\underline{a_{13}a_{21}a_{32}}}$$

$$- \underline{a_{12}a_{21}a_{33}} - \underline{a_{13}a_{22}a_{31}} - \underline{a_{11}a_{23}a_{32}}$$

$$P_{12} = \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix} \qquad P_{13} = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix} \qquad P_{23} = \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix}$$

Determinants: cofactors, inverses & volume

$$\underset{n\times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} \qquad \underset{(n-1)\times(n-1)}{M_{ij}} = \begin{bmatrix} \quad \end{bmatrix}$$

j^{th} column

i^{th} row

remove i^{th} row, j^{th} column from A

ex.

$$\underset{3\times 3}{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

M_{23} M_{11}

$$\underset{2\times 2}{M_{11}} = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} \qquad \underset{2\times 2}{M_{23}} = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

The (i,j) cofactor of $\underset{n\times n}{A}$ is: $\boxed{C_{ij} = (-1)^{i+j}\det(M_{ij})}$

The det formula using cofactors "along row i of A":

$$\boxed{\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}}$$

Often take $i=1$:

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Special case: $n=3$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad (*)\ \det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$A_{3\times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{12} M_{13} M_{11}

(*) $\det(A) = a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \qquad M_{13} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \det(M_{11}) = a_{22}a_{33} - a_{23}a_{32}$$

$$C_{12} = (-1)^{1+2} \det(M_{12}) = -(a_{21}a_{33} - a_{23}a_{31})$$

$$C_{13} = (-1)^{1+3} \det(M_{13}) = a_{21}a_{32} - a_{22}a_{31}$$

(*) $$\det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Example

$$A = \begin{array}{c} \begin{array}{cccc} & a_{12} & & a_{14} \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M_{13} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$$

M_{13} M_{11}

$$\det(A) = a_{11} C_{11} + \cancel{a_{12} C_{12}} + a_{13} C_{13} + \cancel{a_{14} C_{14}}$$

$$C_{11} = (-1)^{1+1} \det(M_{11}) = \det(M_{11}) = 0$$

$$C_{13} = (-1)^{1+3} \det(M_{13}) = \det(M_{13}) = (1)(2)(2) - (2)(1)(1) = 2$$

$C_{11} = (-1)^{1+1} \det(M_{11}) = \det(M_{11}) = 0$

$$C_{13} = (-1)^{1+3} \det(M_{13}) = \det(M_{13}) = (1)(2)(2) - (2)(1)(1) = 2$$

$$\det(A) = \cancel{a_{11} C_{11}} + a_{13} C_{13} = (1)(2) = 2.$$

Determinants help compute inverses

Given $\underset{n \times n}{A}$ recall: $\boxed{A \text{ invertible} \iff \det A \neq 0}$

What's a formula for A^{-1} in terms of A?

Ex. $\underset{2\times 2}{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\det A \neq 0$ then $A^{-1} = \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$\boxed{\text{The } (i,j) \text{ entry of } A^{-1} \text{ is: } \quad C_{ji}/\det A}$$

If $\underset{n\times n}{C}$ is the "cofactor matrix" whose (i,j) entry is C_{ij}

then $$A^{-1} = \frac{1}{\det A} C^T$$

Example

M_{41} M_{31}

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 2 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

M_{21}

$$C_{41} = (-1)^{1+4} \det(M_{41}) = -\det(M_{41}) = -(1-2) = 1$$

$C_{11} = 0$ $\quad C_{12} = 0$ $\quad C_{13} = 2$ $\quad C_{14} = 0$

$C_{21} = 0$ $\quad C_{22} = 4$ $\quad C_{23} = 0$ $\quad C_{24} = -2$

$C_{31} = 0$ $\quad C_{32} = -2$ $\quad C_{33} = 0$ $\quad C_{34} = 2$

$C_{41} = 1$ $\quad C_{42} = 1$ $\quad C_{43} = -1$ $\quad C_{44} = -1$

(others computed similarly)

$$C = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & -2 \\ 0 & -2 & 0 & 2 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

Recall $\det A = 2$

$$A^{-1} = \frac{1}{\det A} C^T = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & -2 & 1 \\ 2 & 0 & 0 & -1 \\ 0 & -2 & 2 & 1 \end{bmatrix}.$$

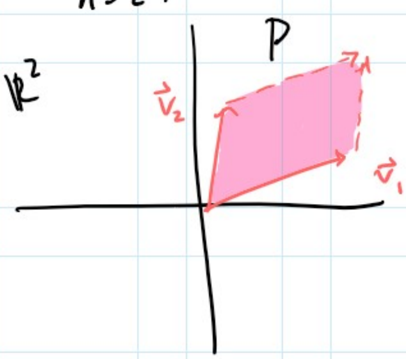
(Check $AA^{-1} = I$!)

Volume

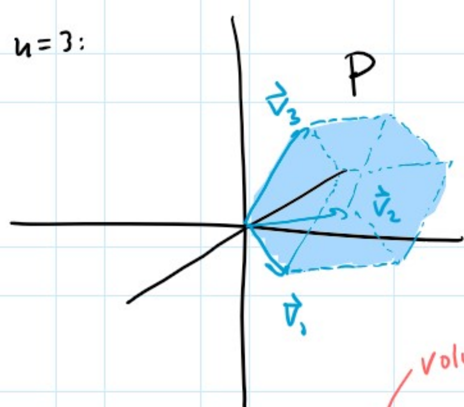
$\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ in $\mathbb{R}^n$

P = n-dimensional parallelepiped in $\mathbb{R}^n$ formed by $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

$n = 2$:



$n = 3$:



volume

Let $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$. Then $\boxed{|\det(A)| = \text{vol}(P).}$

In the 3×3 case, i.e. $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in $\mathbb{R}^3$:

$$(\vec{v}_1 \times \vec{v}_2) \cdot \vec{v}_3 = \det \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$$