Strang's Four Subspaces (i.e. the cover of the textbook!)

Given an $m \times n$ matrix A we can also consider the row space:

$$R(A) = \text{span}(\text{rows of } A) \overset{\text{subspace}}{\subset} \mathbb{R}^n$$

Turns out that
$$\begin{aligned} \dim R(A) &= \#\text{pivot rows of } A \\ &= \#\text{pivot columns of } A \\ &= \dim C(A). \end{aligned}$$

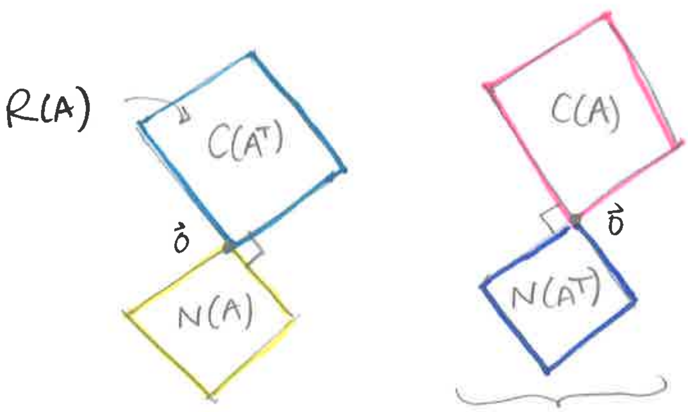
Transpose of an $m \times n$ matrix A is the $n \times m$ matrix gotten by reversing roles of rows/columns. Ex:

$$\underset{3\times 2}{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \underset{2\times 3}{A^T} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

rowspace $R(A^T)$ = column space $C(A)$

column space $C(A^T)$ = row space $R(A)$

Strang's 4 subspaces:



$\mathbb{R}^n$ spanned by $C(A^T)$ & $N(A)$, and they intersect at $\vec{0}$

$\mathbb{R}^m$ spanned by $C(A)$ & $N(A^T)$, and they inters. at $\vec{0}$

Applications of Rank Nullity Theorem

V, W vector spaces $\quad T: V \to W$ linear transformation

inputs $\qquad\qquad$ outputs

$\vec{v}$ vector in V $\xrightarrow{T}$ $T(\vec{v})$ vector in W

Recall that T being a linear transformation means:

- for any $\vec{v}_1, \vec{v}_2$ in V, $\quad T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- for any $\vec{v}$ in V, c any scalar, $\quad T(c\vec{v}) = cT(\vec{v})$

An $m \times n$ matrix A gives a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$.

But the viewpoint is more general.

Given a linear transformation $T: V \to W$ define

$N(T) = \{\vec{v} \text{ in } V \mid T(\vec{v}) = \vec{0}\}$ nullspace of T

$im(T) = \{T(\vec{v}) \text{ in } W \mid \vec{v} \text{ in } V\}$ image of T

"possible outputs" $\qquad$ (also called "range of T")

For T = $m \times n$ matrix A, $im(T) = C(A)$, the column space.

But T may not come to us as a matrix, so there are no "columns" to speak of.

Note $N(T) \overset{\text{subspace}}{\subset} V$ and $im(T) \overset{\text{subspace}}{\subset} W$.

Rank-Nullity Theorem for Linear Transformations:

$$\boxed{\dim(\operatorname{im}(T)) + \dim(N(T)) = \dim(V)}$$

Our goal is to show how useful this theorem is!

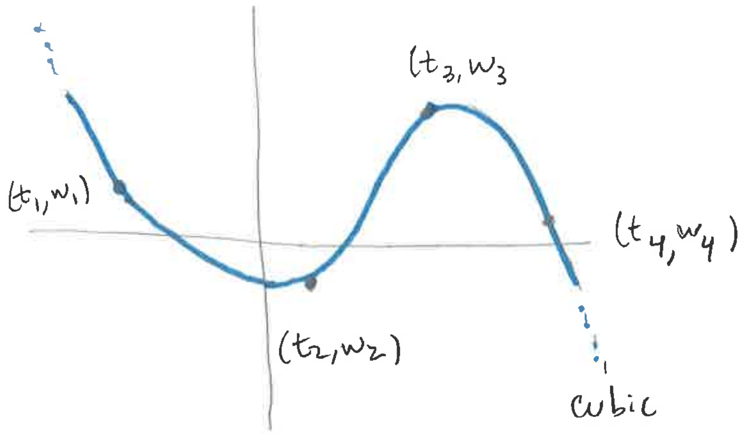
Prescribing values of Polynomials:

A polynomial of degree n is a function $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ where $a_n \neq 0$.

Interpolation problem:

Given distinct real #'s $t_1, t_2, \dots, t_k$ and real #'s $w_1, w_2, \dots, w_k$ can we find a polynomial $p(x)$ such that

$$p(t_i) = w_i \quad \text{for } i = 1, 2, \dots, k \,?$$



Do some examples.
If $k=4$, we can't always use a degree 1 polynomial (a line) or a degree 2 (parabola), but a degree 3 (cubic) will work.

Theorem Let $t_1, \dots, t_{n+1}$ and $w_1, \dots, w_{n+1}$ be real #'s such that the t_i are distinct.

Then there exists a unique polynomial of degree at most n such that $p(t_1) = w_1, \dots, p(t_{n+1}) = w_{n+1}$.

We prove the theorem using Rank-Nullity Thm.

Let $V = \{\text{polynomials of degree} \le n\}$

$$= \{ a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 \mid a_0, a_1, \dots, a_n \in \mathbb{R} \}$$

This has as basis the set $1, x, x^2, \dots, x^n$. So:

$$\dim(V) = n+1.$$

Define $T: V \to \mathbb{R}^{n+1}$ as follows: $T(p(x)) = \begin{bmatrix} p(t_1) \\ p(t_2) \\ \vdots \\ p(t_{n+1}) \end{bmatrix}$ vector in $\mathbb{R}^{n+1}$

input is polynomial $p(x)$ of deg $\le n$

(The #'s $t_1, \dots, t_{n+1}$ and $w_1, \dots, w_{n+1}$ are fixed once and for all.)

Check T is a linear transformation:

- let $p_1(x), p_2(x)$ be in V. We compute

$$T(p_1(x) + p_2(x)) = \begin{bmatrix} p_1(t_1) + p_2(t_1) \\ \vdots \\ p_1(t_{n+1}) + p_2(t_{n+1}) \end{bmatrix}$$

$$= \begin{bmatrix} p_1(t_1) \\ \vdots \\ p_1(t_{n+1}) \end{bmatrix} + \begin{bmatrix} p_2(t_1) \\ \vdots \\ p_2(t_{n+1}) \end{bmatrix} = T(p_1(x)) + T(p_2(x)).$$

- Can similarly check $T(cp(x)) = cT(p(x))$ for c a scalar, $p(x)$ in V.

So indeed T is a linear transformation

That means we can apply the Rank-Nullity Theorem.

Note nullspace $N(T) = \{ p(x) \text{ in } V \mid T(p(x)) = \vec{0} \}$.

But $T(p(x)) = \vec{0}$ means $\begin{bmatrix} p(t_1) \\ \vdots \\ p(t_{n+1}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ i.e. $p(t_i) = 0$ for all $i = 1, \dots, n+1$.

This would mean that $t_1, t_2, \dots, t_{n+1}$ are $n+1$ distinct roots of $p(t)$.

A nonzero degree $\leq n$ polynomial has at most n roots.

So the only possibility is that $p(x)$ is the zero polynomial.

Therefore: $N(T) = \{0\}$ (the zero subspace)

$\rightarrow$ $\dim(N(T)) = 0$.

Rank-Nullity Thm:

$$\dim(\mathrm{im}(T)) + \underbrace{\dim(N(T))}_{0} = \underbrace{\dim(V)}_{n+1}$$

$\rightarrow$ $\dim(\mathrm{im}(T)) = n+1$.

Note $\text{im}(T) = \{\text{all possible outputs of } T\} \subset \mathbb{R}^{n+1}$.

Exercise: if $\mathcal{U}$ is a subspace of W and $\dim \mathcal{U} = \dim W$,

then in fact $\mathcal{U} = W$.

So we get $\text{im}(T) = \mathbb{R}^{n+1}$!

→ everything in $\mathbb{R}^{n+1}$ is an output of T

→ given any $\vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_{n+1} \end{bmatrix}$ in $\mathbb{R}^{n+1}$ there is some $p(x)$ of

degree $\leq n$ such that $T(p(x)) = \vec{w}$, which is to say

$\begin{bmatrix} p(t_1) \\ \vdots \\ p(t_{n+1}) \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_{n+1} \end{bmatrix}$, or $p(t_1) = w_1, \ldots, p(t_{n+1}) = w_{n+1}$.

This is what we wanted to show! ✓

(The "uniqueness" part of the theorem is left as an exercise.)

On to another application.

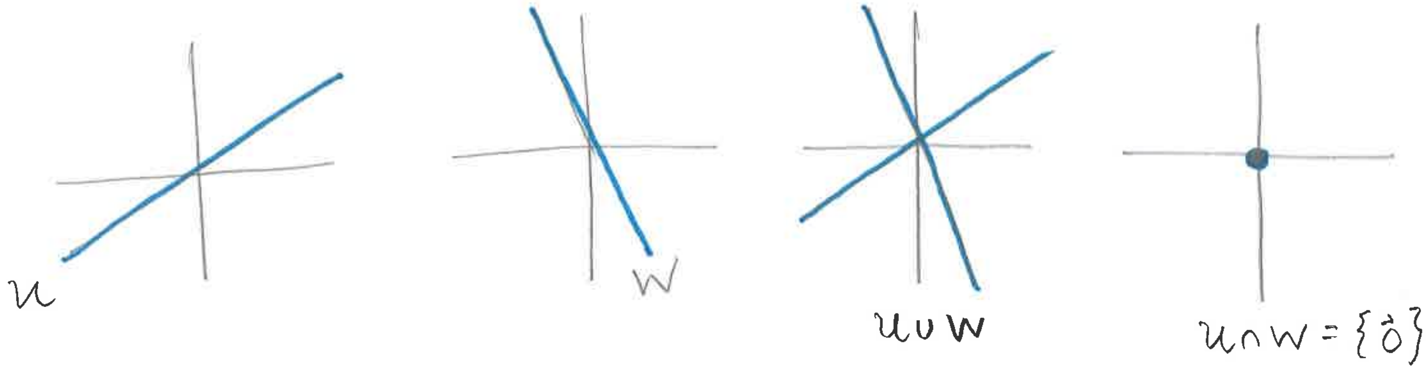
Intersections of subspaces

Let V be a vector space — you can have in mind $V = \mathbb{R}^n$.

Let $\mathcal{U}, W \subset V$ be subspaces. We can define:

Union: $\mathcal{U} \cup W = \{\text{vectors either in } \mathcal{U} \text{ or } W \text{ (or both)}\}$

intersection: $\mathcal{U} \cap W = \{\text{vectors that are both in } \mathcal{U} \text{ \& } W\}$



In general, $\mathcal{U} \cup W$ is not a subspace (see above).

On the other hand, we always have:

$\mathcal{U} \cap W$ is a subspace.

To show this we must verify the conditions of being a subspace.

- $\vec{0}$ is in $\mathcal{U}$ an W since each are subspaces. So $\vec{0}$ is also in $\mathcal{U} \cap W$.
- let $\vec{v}_1, \vec{v}_2$ be vectors in $\mathcal{U} \cap W$.
  So $\vec{v}_1, \vec{v}_2$ are in $\mathcal{U}$ and also W.
  Since $\mathcal{U}$ is a subspace, $\vec{v}_1 + \vec{v}_2$ is in $\mathcal{U}$.
  Since W is a subspace, $\vec{v}_1 + \vec{v}_2$ is in W.
  Therefore $\vec{v}_1 + \vec{v}_2$ is in $\mathcal{U} \cap W$.
- closed under scalar multiplication is similar argument.

Can "fix" $\mathcal{U} \cup W$ so that we get a subspace:

Define $\boxed{\mathcal{U} + W} = \{ \vec{u} + \vec{w} \mid \vec{u} \text{ is in } \mathcal{U},\ \vec{w} \text{ is in } W \}$

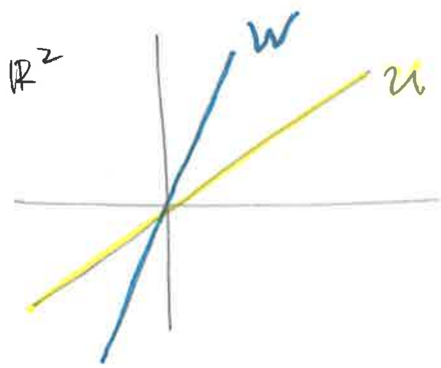
$= $ "$\text{span}(\mathcal{U}, W)$".

$\mathcal{U} + W$ is a subspace. (exercise)

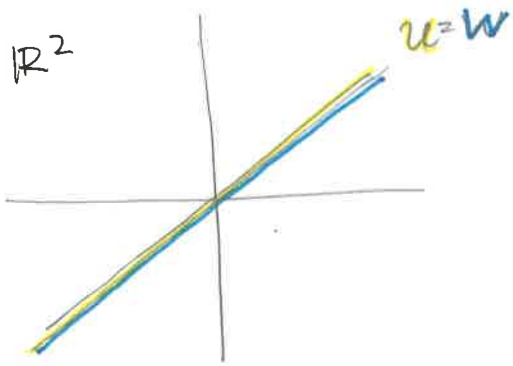
Note $\mathcal{U} \cup W$ is contained inside $\mathcal{U} + W$.

In fact $\mathcal{U} + W$ is the "smallest" subspace containing $\mathcal{U} \cup W$.

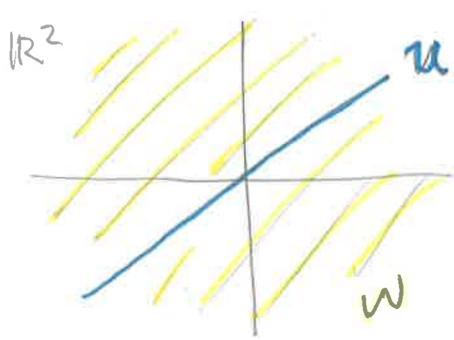
Examples



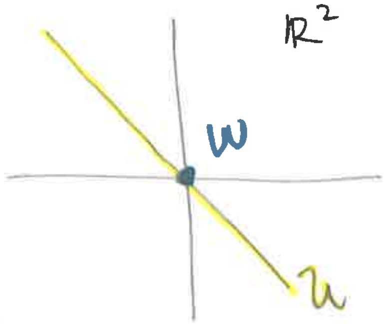
$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U} \cap W$
line	line	$=\mathbb{R}^2$	$\{\vec{0}\}$
dim $=1$	dim $=1$	dim $=2$	dim $=0$



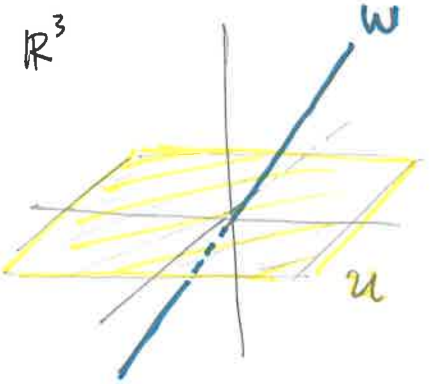
$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U} \cap W$
line	line	$=\mathcal{U}=W$	$=\mathcal{U}=W$
dim $=1$	dim $=1$	dim 1	dim 1



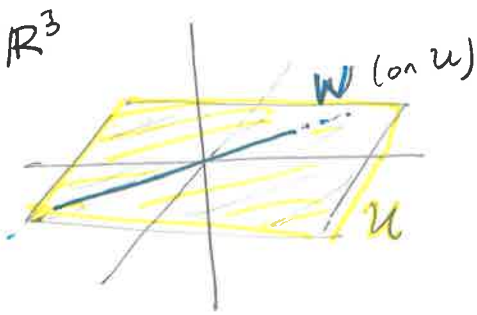
$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U} \cap W$
line	$=\mathbb{R}^2$	$=\mathbb{R}^2$	$=\mathcal{U}$
dim $=1$	dim $=2$	dim $=2$	dim $=1$



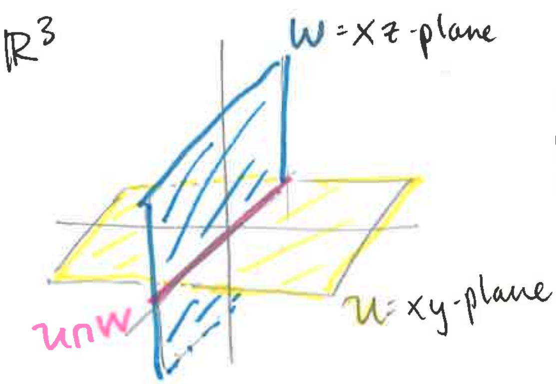
$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U}\cap W$
line	$\{\vec{0}\}$	$=\mathcal{U}$	$=W$
dim $=1$	dim $=0$	dim $=0$	dim $=0$



$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U}\cap W$
plane	line	$=\mathbb{R}^3$	$=\{\vec{0}\}$
dim $=2$	dim $=1$	dim $=3$	dim $=0$



$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U}\cap W$
plane	line	$=\mathcal{U}$	$=W$
dim $=2$	dim $=1$	dim $=2$	dim $=1$



$\mathcal{U}$	W	$\mathcal{U}+W$	$\mathcal{U}\cap W$
plane	plane	$=\mathbb{R}^3$	$= x$-axis
dim $=2$	dim $=2$	dim $=3$	dim $=1$

The pattern is:

$$\dim(\mathcal{U}) + \dim(W) = \dim(\mathcal{U}\cap W) + \dim(\mathcal{U}+W)$$

Next time we'll explain how this formula follows from RN thm.