Dimension

Let V be a vector space. Suppose

$\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$
$\vec{w}_1, \vec{w}_2, \dots, \vec{w}_n$ are two bases of V.

We claim $m = n$. In other words, any two bases of V have the same size.

Proof: As $\vec{u}_1, \dots, \vec{u}_m$ are a basis, we can write each $\vec{w}_i$ in terms of the $\vec{u}_i$'s.

$$(*) \left\{ \begin{array}{l} \vec{w}_1 = a_{11}\vec{u}_1 + \cdots + a_{m1}\vec{u}_m \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ \vec{w}_n = a_{1n}\vec{u}_1 + \cdots + a_{mn}\vec{u}_m \end{array} \right.$$

for some scalars a_{ij}

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}$. Then $(*)$ is: $\underbrace{\begin{bmatrix} | & & | \\ \vec{w}_1 & \cdots & \vec{w}_n \\ | & & | \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_m \\ | & & | \end{bmatrix}}_{\mathcal{U}} A$

i.e. $W = \mathcal{U}A$. Now suppose $n > m$

This means $\#$ cols of A $>$ $\#$ rows of A (A is "short and wide" $[\,\vdots\;\vdots\;\vdots\,]$)

Then the RREF of A $\begin{bmatrix} 1 \cdots & & \\ & 1 \cdots & \\ & & 1 \cdots \end{bmatrix}$ must have some free vars

So there is $\vec{x} \neq \vec{0}$ such that $A\vec{x} = \vec{0}$.

Then $W = \mathcal{U}A \rightarrow W\vec{x} = \mathcal{U}A\vec{x} = \mathcal{U}(\vec{0}) = \vec{0}$

$$\rightarrow x_1 \vec{w}_1 + \dots x_n \vec{w}_n = 0$$

but $\vec{w}_1, \dots, \vec{w}_n$ basis means they are independent,
which implies $x_1 = x_2 = \dots = x_n = 0$, contradicting $\vec{x} \neq \vec{0}$.

The argument for $n < m$ is similar, leading to a contradiction.

Thus it must be that $m = n$. QED

Defn. Let V be a vector space. The dimension of V, written $\dim V$, is the # vectors in any (hence every) basis of V.

Another description of dimension:

$$\boxed{\dim V = \text{minimal \# of vectors needed to span } V}$$

Example $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -5 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -4 \end{bmatrix}$ in $\mathbb{R}^4$

V = subspace of $\mathbb{R}^4$ spanned by these. What is $\dim V$?

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & -5 & -4 \end{bmatrix} \xrightarrow{\text{elim.}} \begin{array}{c} \begin{bmatrix} \boxed{1} & 2 & 1 & 1 \\ 0 & \boxed{1} & -2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & & \end{matrix} \end{array}$$

Basis for V is $\vec{v}_1, \vec{v}_2$. So $\underline{\dim V = 2}$.

Geometrically: V is a 2D plane in $\mathbb{R}^4$ spanned by $\vec{v}_1, \vec{v}_2$

($\vec{v}_3, \vec{v}_4$ are already on the plane)

Example $\dim \mathbb{R}^n = ?$ Intuitively it's n. But we have a rigorous, formal notion of dimension.

Need to check that it gives us the right thing!

It all works out! since $\underbrace{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{n \text{ vectors}}$ is a basis of $\mathbb{R}^n$

(the "standard basis".)

Back to the Rank-Nullity Theorem for $V = \mathbb{R}^m$:

$\vec{v}_1, \dots, \vec{v}_n$ vectors in $\mathbb{R}^m$

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$$ $m \times n$ matrix

column space

$C(A) = \text{span}(\vec{v}_1, \dots, \vec{v}_n) \subset \mathbb{R}^m$

nullspace

$N(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0} \} \subset \mathbb{R}^n$

Rank-Nullity Thm:

$$\boxed{\underbrace{\dim C(A)}_{\text{"rank of } A\text{"}} + \underbrace{\dim N(A)}_{\text{"nullity of } A\text{"}} = \underbrace{n}_{\#\text{ columns of } A}}$$

Example

$$A = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{matrix} \\ \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$n = \#\text{cols of } A = 4$

Do elimination on A:

$$\underbrace{\begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{bmatrix}}_{\text{pivot columns}} \underbrace{\begin{bmatrix} 3/2 & 0 \\ -1/2 & 1 \\ 0 & 0 \end{bmatrix}}_{\text{free columns}}$$

RREF of A

Basis for $C(A)$ is $\vec{v}_1, \vec{v}_2$ so $\dim C(A) = 2$.

What about $N(A)$? Solve $A\vec{x} = \vec{0}$. Free vars $x_3 = s_1$, $x_4 = s_2$

$$x_1 + \frac{3}{2} s_1 = 0$$
$$x_2 - \frac{1}{2} s_1 + s_2 = 0 \quad \longrightarrow \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s_1 \underbrace{\begin{bmatrix} -3/2 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_1} + s_2 \underbrace{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_2}$$
$$0 = 0$$

then $N(A) = \{\text{space of solutions } \vec{x} \text{ to } A\vec{x} = \vec{0}\}$

$= \{ s_1\vec{w}_1 + s_2\vec{w}_2 \mid s_1, s_2 \in \mathbb{R} \} = \text{span}(\vec{w}_1, \vec{w}_2)$.

Can check w_1, w_2 indep. so it's a basis of $N(A)$,

and $\dim N(A) = 2$, i.e. $N(A)$ is a plane in $\mathbb{R}^4$.

In summary, $\dim C(A) + \dim N(A) = n$

$$2 + 2 = 4$$

General case:

let $R = $ RREF of A. Then

$$\boxed{\dim C(A) = \#\text{ vectors in a basis for } C(R) = \#\text{ pivot columns}} \text{ in } R$$

$$\boxed{\dim N(A) = \#\text{ free variables / columns}} \text{ in } R$$

this is b/c the solutions to $A\vec{x} = \vec{0}$ are

$$s_1\vec{w}_1 + s_2\vec{w}_2 + \dots + s_k\vec{w}_k$$

where $s_1, \to s_k$ are the free vars.

So for $V = \mathbb{R}^m$ the Rank-Nullity Thm boils down to:

$$\boxed{\underbrace{\dim C(A)}_{\#\text{pivot columns in RREF}} + \underbrace{\dim N(A)}_{\#\text{free columns in RREF}} = \underbrace{n}_{\#\text{columns in } A}}$$

which is clearly true!

Warning: Although $\dim C(A) = \#\text{pivot columns in } C(R)$

$= \dim C(R)$ (where R = RREF of A)

it is usually not true that $C(A) = C(R)$.

Only the dimensions are the same, not the spaces!

ex: $A = \begin{bmatrix} 1 & -1 & 2 & -1 \\ 0 & 2 & -1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ $\qquad \underset{\text{RREF}}{R} = \begin{bmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right) = \text{plane in } \mathbb{R}^3 \ \left\{ s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \,\middle|\, s,t \in \mathbb{R} \right\}$

$C(R) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = xy\text{-plane}.$

So in this example, $C(A) \neq C(R)$ (note $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ not in xy-plane).

On the other hand, $N(A) = N(R)$ always!

(nullspace of A = null space of RREF of A)