Independence, Bases

MTH 210

Recall: vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in a vector space V are (linearly) independent if the only linear combination

(*) $$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n \qquad (x_1, \ldots, x_n \text{ scalars})$$

that is $\vec{0}$ is the one where $x_1 = x_2 = \cdots = x_n = 0$.
Otherwise they're dependent.

Given $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^m$ define the matrix

$$A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix} \qquad m \times n \text{ matrix}$$

Then (*) is equal to $A\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$. We obtain:

$$\boxed{\vec{v}_1, \ldots, \vec{v}_n \text{ are independent} \iff \text{only soln. to } A\vec{x} = \vec{0} \text{ is } \vec{x} = \vec{0}}$$

We can phrase this in terms of the nullspace of A as follows. recall $N(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$. Then:

$$\boxed{\vec{v}_1, \ldots, \vec{v}_n \text{ are independent} \iff N(A) = \{\vec{0}\}}$$

We have an algorithm to determine when $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^m$ are independent: Apply elimination to $A\vec{x} = \vec{0}$ (with A as above) to solve for $\vec{x}$. If the only solution is $\vec{x} = \vec{0}$, then they are independent (otherwise, dependent).

Example Determine if $\begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ are independent.

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 2 \end{bmatrix}$. Solve $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \xrightarrow{\text{do elimination}} \cdots \longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 2 & 0 \end{array}\right] \\ \begin{array}{ccc} x & y & z \end{array} \end{array}$$

free var is $z = t$. get: $y + 2t = 0$

$\rightarrow y = -2t$

$x + 2t = 0 \rightarrow x = -2t$

Solns. are $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$. (a line through $\vec{0}$ in $\mathbb{R}^3$).

$\vec{0}$ is not the only soln, so the three vectors are dependent.

Theorem If $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^m$ and

$$n > m$$

then these vectors are dependent.

Proof: Let $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$ & apply elim. to $A\vec{x} = \vec{0}$.

If $\vec{v}_1, \ldots, \vec{v}_n$ are independent, elimination produces no free vars (since these would indicate more solns than just $\vec{0} = \vec{x}$).

Look at possible RREF's with no free vars:

$$\begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 0 \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \text{ if } m = n \qquad \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & 1 & 0 & \cdots \\ 0 & & & 1 \\ & & & 0 \\ & & & 0 \end{bmatrix} \text{ if } m > n$$

If $m < n$ it's impossible to have no free variables! QED.

Recall for $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in V the span is a subspace defined by

$$\text{span}(\vec{v}_1, \ldots, \vec{v}_n) = \{ x_1\vec{v}_1 + \ldots + x_n\vec{v}_n \mid x_1, \ldots, x_n \in \mathbb{R} \}$$

We say $\vec{v}_1, \ldots, \vec{v}_n$ span a subspace $W \subset V$ if

$$W = \text{span}(\vec{v}_1, \ldots, \vec{v}_n).$$

In words: $\vec{v}_1, \ldots, \vec{v}_n$ span W if every vector in W can be written as

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$$

for some $x_1, \ldots, x_n$ scalars.

We now come to one of the most important notions in Linear Algebra.

Defn. A **basis** of a vector space is a collection of vectors $\vec{v}_1, \ldots, \vec{v}_n$ such that

(i) $\vec{v}_1, \ldots, \vec{v}_n$ span the vector space

(ii) $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent

Thm $\vec{v}_1, \ldots, \vec{v}_n$ is a basis of V $\iff$ any vector $\vec{v}$ in V can be written as

$$(*) \qquad x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$$

for exactly one set of scalars $x_1, x_2, \ldots, x_n$.

Proof: By (i) in definition of basis, we know we can write any $\vec{v}$ in the form $(*)$ for some $x_1, \ldots, x_n$.

Suppose $\vec{v} = y_1\vec{v}_1 + \cdots + y_n\vec{v}_n$ where $y_1, \ldots, y_n$ are scalars, possibly different from the x_i. Then:

$$\vec{0} = \vec{v} - \vec{v} = (x_1\vec{v}_1 + \ldots + x_n\vec{v}_n) - (y_1\vec{v}_1 + \ldots + y_n\vec{v}_n)$$
$$= (x_1 - y_1)\vec{v}_1 + (x_2 - y_2)\vec{v}_2 + \ldots + (x_n - y_n)\vec{v}_n$$

Now using (ii) (that $\vec{v}_1, \ldots, \vec{v}_n$) are linearly indep)

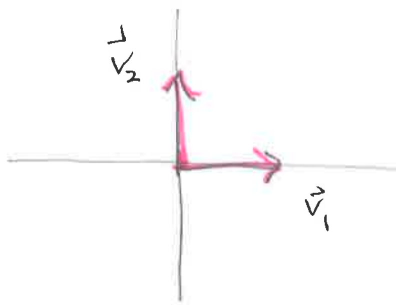
we obtain $x_1 - y_1 = 0,\ x_2 - y_2 = 0, \ldots \rightarrow x_1 = y_1,\ x_2 = y_2, \ldots$

Thus as claimed, $\vec{v}$ can be written as (*)

for a unique set of scalars $x_1, \ldots, x_n$. QED

Examples

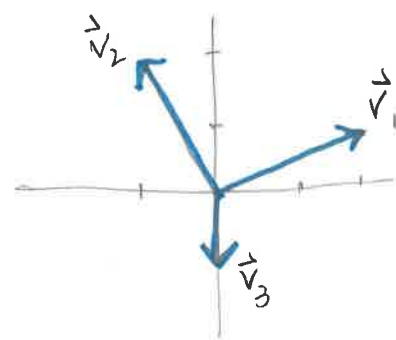
(1) Is $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ a basis of $\mathbb{R}^2$?



Yes. (i) $\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$ ✓

(ii) $\vec{v}_1, \vec{v}_2$ independent ✓

(2) Let $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Is $\vec{v}_1$ a basis of $\mathbb{R}^2$?

(i) $\vec{v}_1$ does not span $\mathbb{R}^2$ ✗

(ii) $\vec{v}_1$ is independent ✓

No.

However note $\vec{v}_1$ is a basis of the line $\text{span}(\vec{v}_1)$.

Is $\vec{v}_1, \vec{v}_2, \vec{v}_3$ a basis of $\mathbb{R}^2$?

No. They are not independent (since $3 > 2$).

Is $\vec{v}_1, \vec{v}_2$ a basis of $\mathbb{R}^2$? Yes

(i) $\vec{v}_1, \vec{v}_2$ span $\mathbb{R}^2$ ✓ (e.g. $A = [\vec{v}_1 \; \vec{v}_2]$ invertible)

(ii) $\vec{v}_1, \vec{v}_2$ independent ✓

By the previous theorem, we are able to write any vector in $\mathbb{R}^2$ uniquely in terms of $\vec{v}_1, \vec{v}_2$..

For example, if $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ we will find that

$$\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \left(\tfrac{4}{5}\right)\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \left(-\tfrac{7}{5}\right)\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\vec{v} = \tfrac{4}{5}\vec{v}_1 + -\tfrac{7}{5}\vec{v}_2$$

and $4/5, -7/5$ are the only scalars that make this work.

(Basis $\leftrightarrow$ "coordinate system")

Algorithm to test if $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is a basis of $\mathbb{R}^m$:

if $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{bmatrix}$ then

$$\boxed{\begin{gathered} \vec{v}_1, \dots, \vec{v}_n \text{ is a basis of } \mathbb{R}^m \\ \Updownarrow \\ n = m \text{ and } A \text{ is invertible} \end{gathered}}$$

...so we can use elimination.

What if we're interested in a vector space $\neq \mathbb{R}^m$?

(e.g. V = subspace of $\mathbb{R}^m$ such as a plane through $\vec{0}$, ...)

Let $\vec{v}_1, \ldots, \vec{v}_n$ be vectors in $\mathbb{R}^m$ that span a subspace $V \subset \mathbb{R}^m$:

$$V = \text{span}(\vec{v}_1, \ldots, \vec{v}_n).$$

How can we tell if $\vec{v}_1, \ldots, \vec{v}_n$ are a basis of V?

Need to test if they are independent.

Let $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \xrightarrow{\text{elimination}} \begin{bmatrix} \boxed{1} & \cdots & 0 & \cdots \\ 0 & 0 & \boxed{1} & \cdots \\ \vdots & & 0 & \end{bmatrix}$ RREF of A

(Echelon form suffices)

pivot columns (other columns are free)

> The collection of $\vec{v}_i$'s corresponding to pivot columns gives a basis for $V = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$

Ex.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 5 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{in } \mathbb{R}^4.$$

Find a basis for

$V = \text{span}(\vec{v}_1, \ldots, \vec{v}_4)$.

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ -2 & 1 & -1 & 0 \end{bmatrix} \xrightarrow{\text{elim.}} \begin{array}{c} \begin{bmatrix} \boxed{1} & 2 & 1 & 0 \\ 0 & \boxed{5} & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{-1} \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \qquad\quad \vec{v}_4 \end{array}$$

$\ldots \to \vec{v}_1, \vec{v}_2, \vec{v}_4$ basis for V.