Rank-Nullity Theorem

MTH 210

Let A be an $m \times n$ matrix. Recall we have two important vector spaces* associated to A:

column space $\quad C(A) = \text{span}(\text{col}_1, \ldots, \text{col}_n) \overset{\text{subspace}}{\subset} \mathbb{R}^m$

null space $\quad N(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \text{ solving } A\vec{x} = \vec{0} \} \subset \mathbb{R}^n$

The new viewpoint we want to emphasize:

Let V, W be vector spaces. A linear transformation

$$T : V \to W$$

from V to W is an assignment that takes each vector $\vec{v}$ in V to a vector $T(\vec{v})$ in W that satisfies:

- $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$
- $T(c\vec{v}) = c\,T(\vec{v})$

for vectors $\vec{v}, \vec{v}_1, \vec{v}_2$ in V and scalars c.

An $m \times n$ matrix defines a linear transformation:

$$A : \mathbb{R}^n \to \mathbb{R}^m$$

input vector $\vec{x}$ $\rightsquigarrow$ vector $A\vec{x}$ output

(*: a subspace of a vector space is itself a vector space!)

$C(A) = \{$possible outputs $A\vec{x}$ in $\mathbb{R}^m\}$

$N(A) = \{$possible inputs $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{0}.\}$

Warning: $C(A)$, $N(A)$ are generally subspaces of diff. vector spaces!

Examples

(0) A = zero $m\times n$ matrix $\qquad A: \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \rightsquigarrow \vec{0}$ (regardless of $\vec{x}$)

$C(A)$ = possible outputs $= \{\vec{0}\}$

$N(A) = \{\vec{x}$ such that $A\vec{x} = \vec{0}\} =$ all $\vec{x}$! $= \mathbb{R}^n$.

$$\underset{0}{\dim C(A)} + \underset{n}{\dim N(A)} = n$$

(Still haven't defined "dimension" = dim rigorously – coming soon!)

(1) A invertible $n\times n$ matrix

$C(A) = \mathbb{R}^n$ (explained last lecture)

$N(A) = \{$solutions to $A\vec{x} = \vec{0}\} = ?$

Recall A invertible implies $A^{-1}A\vec{x} = A^{-1}\vec{0} \to \vec{x} = \vec{0}$.

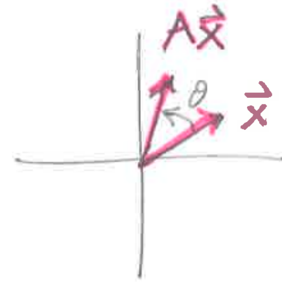
i.e. only solution is $\vec{0}$. So $N(A) = \{\vec{0}\}$.

$$\underset{n}{\dim C(A)} + \underset{0}{\dim N(A)} = n$$

(2) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$

rotates vectors ccw by θ



$C(A) = \mathbb{R}^2$

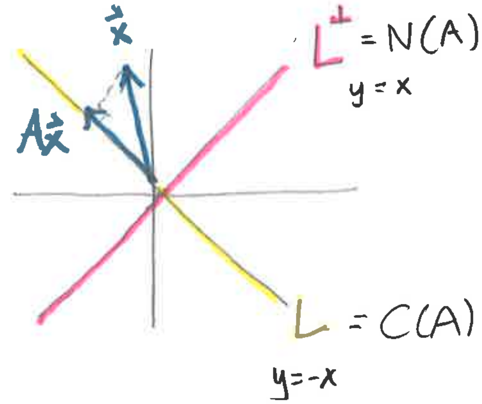
$N(A) = \{\vec{0}\}$

$\underset{2}{\dim C(A)} + \underset{0}{\dim N(A)} = 2$ ($= n$ here)

(3) $A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$

projects vectors onto L



$C(A) = L$ (last time) $\subset \mathbb{R}^2$

$N(A) = \{\vec{x} \text{ in } \mathbb{R}^2 \mid A\vec{x} = \vec{0}\}$

$= \{\vec{x} \text{ in } \mathbb{R}^2 \mid \vec{x} \perp L\} = L^\perp$, the line $y = x$ perp. to L

$\underset{1}{\dim C(A)} + \underset{1}{\dim N(A)} = 2$ ($= n$ here)

(4) $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$

$A : \mathbb{R}^3 \to \mathbb{R}^3$

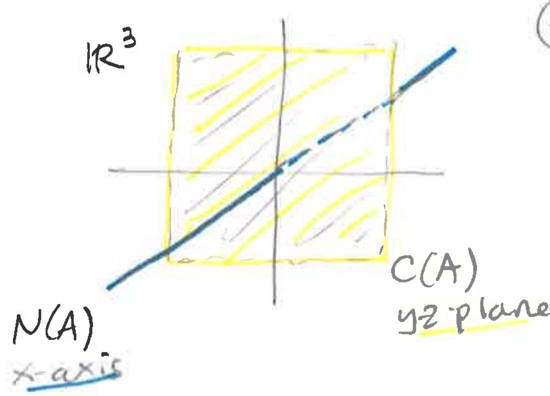
Projects $\vec{x}$ onto yz-plane then rotates CCW by θ in yz-plane

$C(A) = yz\text{-plane} \subset \mathbb{R}^3$

$N(A) = \{\vec{x} \text{ such that } A\vec{x} = \vec{0}\}$

$= \{\vec{x} \perp yz\text{-plane}\} = x\text{-axis} \subset \mathbb{R}^3.$

$$\dim C(A) + \dim N(A) = 3 \quad (=n)$$
$$\quad 2 \qquad\qquad\quad 1$$



See a pattern?

Rank-Nullity Theorem:

(or: "the most important result in linear algebra")

A $m \times n$ matrix

$$\boxed{\dim C(A) + \dim N(A) = n}$$

rank = $\dim C(A)$ = "rank of A"

nullity = $\dim N(A)$ = "nullity of A"

n = # columns of A

So theorem says: $\boxed{\text{rank} + \text{nullity} = n}$

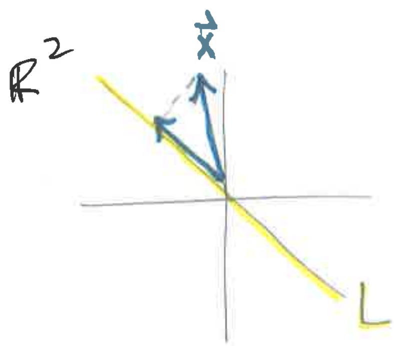
Note n is also the dimension of all possible "inputs".

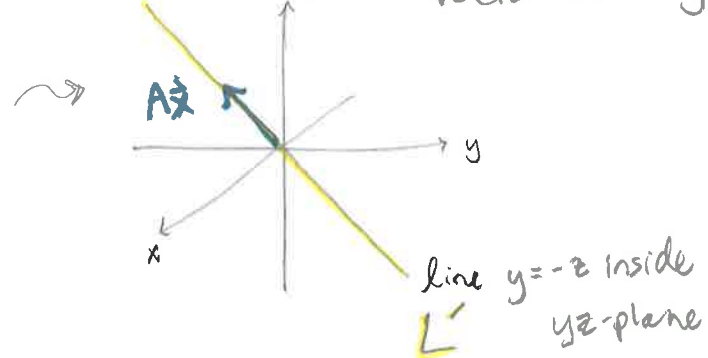
Example

$$A = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$A: \mathbb{R}^2 \to \mathbb{R}^3$

A projects onto line $L = \{y = -x\}$ then includes vector into yz-plane





$\underbrace{C(A) = L'}_{\text{line}} \subset \mathbb{R}^3$ $\qquad$ $\underbrace{N(A) = L^\perp}_{\substack{\text{line} \\ y=x}} \subset \mathbb{R}^2$

$$\underset{1}{\dim C(A)} + \underset{1}{\dim N(A)} = 2 \ (= n!)$$

We'll come back to this awesome theorem later.

To really make sense of dimension we need:

Linear Independence

V vector space, for example $V = \mathbb{R}^m$ or $V = C(A)$ or $V = N(A)$ of a matrix.

A set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in V is (linearly) independent if the only solution to

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = \vec{0}$$

is $x_1 = x_2 = \cdots = x_n = 0$. ($x_i$ are scalars)

Otherwise, $\vec{v}_1, \ldots, \vec{v}_n$ are (linearly) dependent.

So: $\vec{v}_1, \ldots, \vec{v}_n$ are independent if only linear comb. of them that gives $\vec{0}$ is the comb. $0\vec{v}_1 + \cdots + 0\vec{v}_n$.

Examples

(1) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ in $\mathbb{R}^2$

Possible x_1, x_2 such that

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}?$$

$$\rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \rightarrow x_1 = x_2 = 0.$$

So these two vectors are independent.

(2) Similarly $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ independent in $\mathbb{R}^3$.

pattern continues...

$$\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$$ independent in $\mathbb{R}^n$

(3) $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

Possible x_1, x_2, x_3

$$x_1\begin{bmatrix}1\\1\end{bmatrix} + x_2\begin{bmatrix}1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}?$$

$x_1 = 1$, $x_2 = -1$, $x_3 = -1$ is a non-zero solution.

So these vectors are dependent.

(4) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}$

$$x_1\begin{bmatrix}1\\0\end{bmatrix} + x_2\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$x_1 = 0 \quad x_2 = 1 \longrightarrow$ dependent.

[If $\vec{v}_1, \ldots, \vec{v}_n$ has $\vec{v}_i = \vec{0}$ for one of the vectors, they're dependent.]