Spans & Column Spaces & Nullspaces

Last time: vector spaces & subspaces

Given vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \ldots$ in a vector space V

Most likely, these do not constitute a subspace of V

We look for the "smallest" subspace containing $\vec{u}_1, \vec{u}_2 \ldots$

Defn: $\boxed{\text{Span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n) = \{ c_1\vec{u}_1 + c_2\vec{u}_2 + \ldots + c_n\vec{u}_n \mid c_1, \ldots, c_n \in \mathbb{R} \}}$

You can verify that this is a subspace.

Examples

(0) $\text{Span}(\vec{0}) = \{\vec{0}\}$

(1) if $\vec{u}$ is a non-zero vector in $\mathbb{R}^n$

then $\text{span}(\vec{u}) = \{ t\vec{u} \mid t \in \mathbb{R} \}$ is a line.

(2) $\vec{u}, \vec{v}$ nonzero vectors in $\mathbb{R}^n$. Possibilities for $\text{span}(\vec{u}, \vec{v})$?

- plane through $\vec{0}$ $(n \geq 2)$
- line through $\vec{0}$

Second case occurs if $\vec{u} = c\vec{v}$ for some scalar $c \in \mathbb{R}$.

(3) $\text{span}(\vec{u}, \vec{v}, \vec{w})$ is either: $\{\vec{0}\}$, line through $\vec{0}$, plane through $\vec{0}$, 3D subspace

Very important subspace associated to a matrix is its column space.

A $m\times n$ matrix $\qquad A = \begin{bmatrix} | & | & & | \\ col_1 & col_2 & \cdots & col_n \\ | & | & & | \end{bmatrix}$

Column space of A is $\boxed{C(A) = \text{span}(col_1, col_2, \ldots, col_n)}$.

It's a subspace of $\mathbb{R}^m$!

Example

$$A = \begin{array}{ccc} col_1 & col_2 & col_3 \end{array} \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad C(A) = \left\{ x_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \,\middle|\, x_1, x_2, x_3 \in \mathbb{R} \right\}$$

$$\left(= \text{span}(col_1, col_2, col_3) \right)$$

Is $\vec{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in the column space $C(A)$?

This is just the "column picture" version of asking to solve the system $A\vec{x} = \vec{b}$.

(Answer is "yes": for example, $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ works.)

In general: (*) $\boxed{\vec{b} \text{ is in the column space } C(A)} \Leftrightarrow \boxed{A\vec{x} = \vec{b} \text{ has a solution in } \vec{x}}$

Thus we can write

$$\boxed{C(A) = \{ A\vec{x} \mid \vec{x} \text{ vector in } \mathbb{R}^n \}}$$

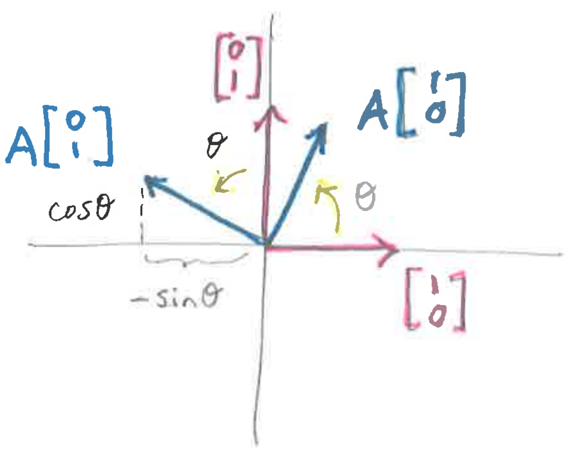
$C(A)$ is also sometimes called the image of A.

Example

(1) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ θ any real #

$$A\vec{x} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta x - \sin\theta y \\ \sin\theta x + \cos\theta y \end{bmatrix}$$

In particular, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$



matrix A rotates by angle θ in counterclockwise dir

What is $C(A)$?

Is A invertible?

Yes, since $\det(A) = \det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta = 1 \neq 0$.

A invertible says $A\vec{x} = \vec{b}$ always solvable ($\vec{x} = A^{-1}\vec{b}$)

So by (*), $C(A) = \mathbb{R}^2$ (the whole space).

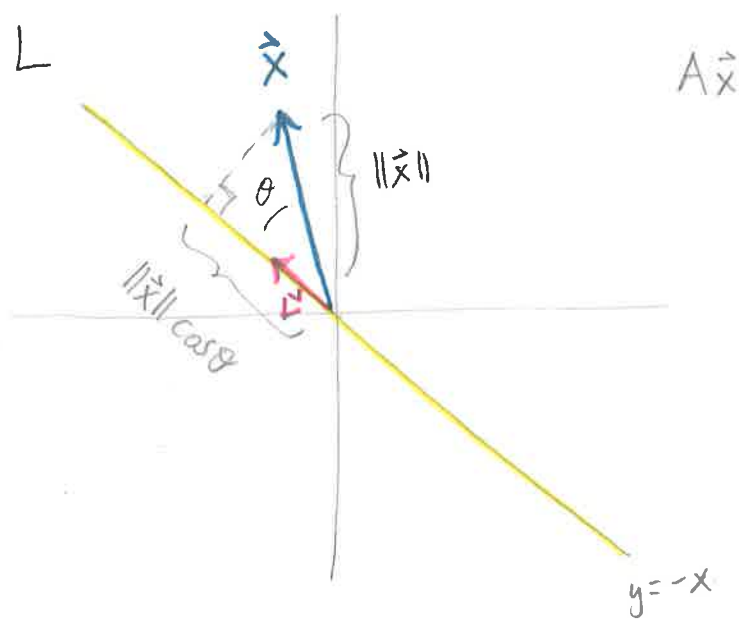
Geometric viewpoint: every vector is the rotation of some other vector by angle θ

More generally, $\boxed{\text{if } A \text{ is an invertible } n\times n \text{ matrix, then } C(A) = \mathbb{R}^n \text{ (whole space).}}$

(Same reasoning, using (*).)

(2) $A = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$. What is $C(A)$?

Claim: A projects vectors in $\mathbb{R}^2$ to the line $L = \{ y = -x \}$.



$A\vec{x} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x-y) \\ -\frac{1}{2}(x-y) \end{bmatrix}$

$\vec{v}$ unit vector on L

$\vec{v} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

Now use $\|\vec{x}\| \underbrace{\|\vec{v}\|}_{=1} \cos\theta = \vec{x} \cdot \vec{v}$

$\rightarrow \|\vec{x}\| \cos\theta = \vec{x} \cdot \vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = -\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y$

Projection of $\vec{x}$ onto L is

$$(\|\vec{x}\| \cos\theta)\,\vec{v} = \left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)\begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}(x-y) \\ -\frac{1}{2}(x-y) \end{bmatrix} = A\vec{x},$$

So the claim is justified.

Back to $C(A)$:

Geometric: A projects vectors to L

so $C(A) = \{A\vec{x} \mid \vec{x} \text{ in } \mathbb{R}^2\} = L \rightarrow \boxed{C(A) = L}$

Algebraic: $C(A) = \text{span}(\text{col}_1, \text{col}_2) = \text{span}\left(\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}\right)$

$$= \text{span}\left(\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}\right) = \left\{ t\begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix} \,\middle|\, t \in \mathbb{R} \right\} = L.$$

(3)

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

A takes a vector $\vec{x}$ in $\mathbb{R}^3$ to $A\vec{x}$ in $\mathbb{R}^3$, which is first projection onto yz-plane then rotation by θ (CCW) in yz-plane.

Can then see that $C(A) =$ yz-plane.

Nullspaces:

A $m \times n$ matrix. The nullspace of A is defined as:

$$\boxed{N(A) = \{ \vec{x} \text{ in } \mathbb{R}^n \mid A\vec{x} = \vec{0} \}}$$

i.e, $N(A)$ = set of solutions to $A\vec{x} = \vec{0}$.

$N(A)$ is sometimes also called the kernel of A.

Claim: $N(A)$ is a subspace of $\mathbb{R}^n$.

- check $\vec{0}$ is in $N(A)$: $A\vec{0} = \vec{0}$, so indeed $\vec{0}$ is in the nullspace

- if $\vec{u}, \vec{v}$ are in $N(A)$, is $\vec{u} + \vec{v}$ in $N(A)$?

  $\vec{u}$ in $N(A)$ means $A\vec{u} = \vec{0}$

  $\vec{v}$ in $N(A)$ means $A\vec{v} = \vec{0}$

  combine: $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$.

  finally, $A(\vec{u}+\vec{v}) = \vec{0}$ says that $\vec{u}+\vec{v}$ is in $N(A)$.

- if $\vec{u}$ is in $N(A)$, is $c\vec{u}$ in $N(A)$ for scalars c?

  $\vec{u}$ in $N(A)$ means $A\vec{u} = \vec{0}$

  then $A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$ implies $c\vec{u}$ is in $N(A)$.