Vector Spaces & Subspaces

Recall properties satisfied by vectors & scalars:

(1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$

(3) $\vec{u} + \vec{0} = \vec{u}$

(4) $\vec{u} + (-\vec{u}) = \vec{0}$

(5) $1\vec{u} = \vec{u}$

(6) $(cd)\vec{u} = c(d\vec{u})$

(7) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

(8) $(c+d)\vec{u} = c\vec{u} + d\vec{v}$

$\vec{u}, \vec{v}, \vec{w}$ any vectors

c, d scalars

We abstract these into the following concept.

A vector space is a set V of "vectors" together with two operations:

$$\underset{\text{vector}}{\vec{u}}, \underset{\text{vector}}{\vec{v}} \xrightarrow[\text{vector addition}]{\text{"}+\text{"}} \text{"}\vec{u}+\vec{v}\text{"}$$

$$\underset{\text{scalar}}{c}, \underset{\text{vector}}{\vec{v}} \xrightarrow[\text{multiplication}]{\text{scalar}} \text{"}c\vec{v}\text{"}$$

There is a unique "zero vector" $\vec{0}$ in V.

We require axioms (1)–(8) above to hold for V.

$\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3, \ldots$ are vector spaces, of course!

$$\mathbb{R}^n = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \;\middle|\; u_1, \ldots, u_n \in \mathbb{R} \right\}$$

But there are many other vector spaces!

Examples

(1) $M = \{ m\times n \text{ matrices} \}$, set of all $m\times n$ matrices,

"$+$" is usual matrix addition, scalar mult. as well

"$\vec{0}$" is the $m\times n$ matrix with all zeros.

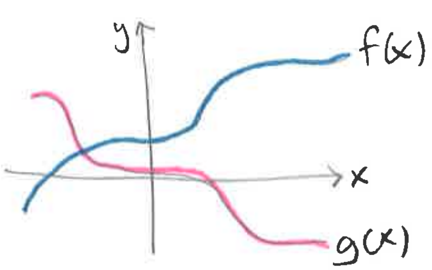
ex. the 2×2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a "vector" in M

$$\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = (2)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

So the "vector" $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ is a linear combination

of the "vectors" $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

(2) $F = \{ \text{real-valued functions } f(x) \}$

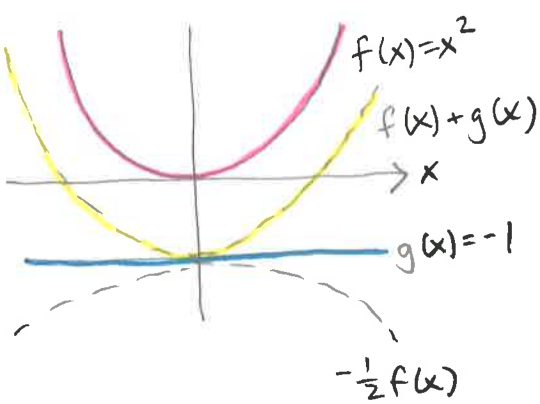


A "vector" here is a function $f = f(x)$.

"$+$" given by: $(f+g)(x) = f(x) + g(x)$

c scalar, then $(cf)(x) = cf(x)$.

"$\vec{0}$" is the constant zero function.



Then, with all this said, F is a vector space.

(This is an example of an "∞-dimensional" vectorspace.)

(3) $V = \mathbb{R}^2$, but let's change "+".

Given $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ define

$$\text{"}\vec{u}+\vec{v}\text{"} = \begin{bmatrix} u_1 + v_2 \\ u_2 + v_1 \end{bmatrix}.$$

Scalar multiplication: the usual.

Claim: This is not a vector space.

Axiom (1): $\vec{u}+\vec{v} = \vec{v}+\vec{u}$. Is it true here?

$$\text{"}\vec{v}+\vec{u}\text{"} = \begin{bmatrix} v_1 + u_2 \\ v_2 + u_1 \end{bmatrix}.$$

So if $\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ we get:

$$\text{"}\vec{u}+\vec{v}\text{"} = \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+1 \end{bmatrix} = \text{"}\vec{v}+\vec{u}\text{"}.$$

So Axiom (1) fails. So not a vector space!

A subspace W of a vector space V is a collection of vectors in V (a subset of V) such that:

- $\vec{0}$ is in W.
- if $\vec{u}$ & $\vec{v}$ are in W then $\vec{u}+\vec{v}$ is in W.
- if $\vec{u}$ is in W and $c \in \mathbb{R}$ then $c\vec{u}$ is in W.

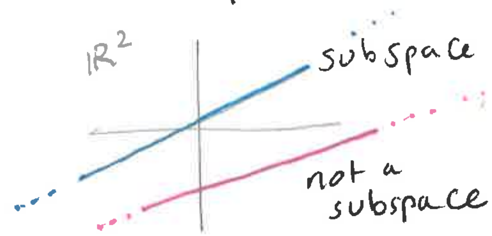
(So W is "closed under addition and scaling of vectors".)

If W is a subspace and $\vec{u}, \vec{v}$ are in W then any linear combination $c\vec{u}+d\vec{v}$ is in W.

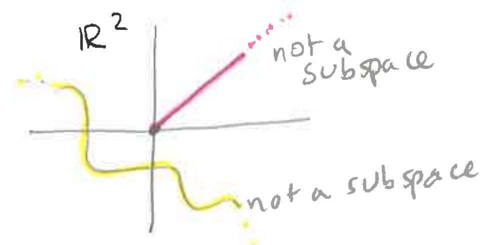
Examples

(0) $W=\{\vec{0}\}$ is a subspace of any vector space

(1) Possible subspaces of $\mathbb{R}^2$:

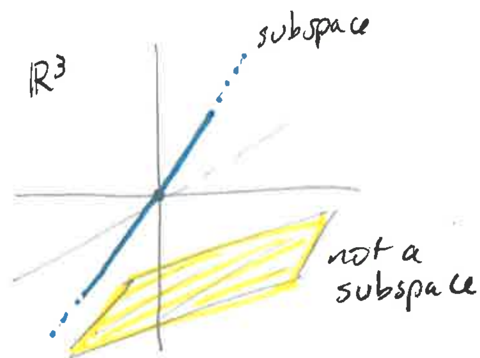


- $\{\vec{0}\}$
- a line through $\vec{0}$
- all of $\mathbb{R}^2$



(2) Possible subspaces of $\mathbb{R}^3$:

- $\{\vec{0}\}$
- a line through $\vec{0}$
- a plane through $\vec{0}$
- all of $\mathbb{R}^3$



(3) Consider the following set W of vectors in $\mathbb{R}^2 = V$:

$$W=\left\{\vec{u}=\begin{bmatrix}u_1\\u_2\end{bmatrix} \,\middle|\, u_1 \geq u_2\right\}$$

Is this a subspace of V?

Take $\vec{u}=\begin{bmatrix}1\\0\end{bmatrix}$. This is in W. Multiply by (-1): $\begin{bmatrix}-1\\0\end{bmatrix}$.

But $\begin{bmatrix}-1\\0\end{bmatrix}$ is not in W. So W is not a subspace.

(4) Let W be the set of vectors in $V = \mathbb{R}^2$ as follows:

$$W = \left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \middle| \; u_1 + u_2 = 0 \right\}$$

Is W a subspace of V? Check conditions:

- $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in W? $0+0=0$, so yes!
- if $\vec{u}, \vec{v}$ in W is $\vec{u}+\vec{v}$ in W?

  Say $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are in W. Then $u_1+u_2=0$, $v_1+v_2=0$.

  So $\vec{u}+\vec{v} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$ satisfies $(u_1+v_1)+(u_2+v_2)=0$. Yes!
- if $\vec{u}$ in W, and c = scalar, is $c\vec{u}$ in W?

  Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ so that $u_1+u_2=0$.

  Then $c(u_1+u_2)=0$, or $cu_1+cu_2=0$.

  This means $c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$ is in W. So Yes!

W passes all conditions to be a subspace, so it's a subspace.

Alternatively:

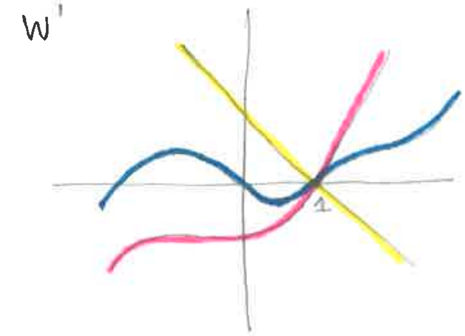
W is the subset of $\mathbb{R}^2$ defined by eq. $x+y=0$ so it is a line through the origin, and I told you that this is a subspace.

But the "correct" (or "direct") argument is the previous one.

(5) Recall the vector space $F = \{\text{real-valued functions } f(x)\}$

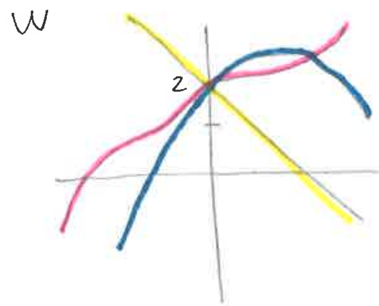
Consider

$$W = \{ f(x) \mid f(1) = 0 \}$$



and

$$W' = \{ f(x) \mid f(0) = 2 \}$$



Q: Which of W, W' is a subspace of the vector space F?

Note "$\vec{0}$", the constant zero function, is not in W'!
Just for this reason, W' is not a subspace.

We can show W is a subspace:

- "$\vec{0}$" is in W since the zero function evaluates to 0 at $x=1$
- suppose $f(x), g(x)$ in W.
  then $f(1)=0$, $g(1)=0$.
  $f+g$ satisfies $(f+g)(1) = f(1)+g(1) = 0$.
  So $f+g$ is in W.
- suppose $f(x)$ is in W, $c \in \mathbb{R}$
  then $f(1)=0$.
  cf satisfies $(cf)(1) = cf(1) = 0$, so is also in W.

W passes all the criteria to be a subspace.