LU decomposition of matrices

MTH 210 2/23/23

Let's start with a matrix A.

Suppose we do elimination to get A into Echelon form.

(not necessarily to RREF!)

Ex.

$$A = \begin{bmatrix} \boxed{1} & 2 & -1 \\ -1 & -1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{E_{21,1} \\ \text{(elimination matrix)}}]{} \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow[E_{31,-2}]{} \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{1} & 0 \\ 0 & -3 & 2 \end{bmatrix} \xrightarrow[E_{32,3}]{}$$

$$\mathcal{U} = \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{2} \end{bmatrix}$$

Echelon form

As A is square, the Echelon form $\mathcal{U}$ is an upper triangular matrix (zeros below diagonal).

The above process is encoded by the relation:

$$E_{32,3}\, E_{31,-2}\, E_{21,1}\, A = \mathcal{U} \qquad (*)$$

Multiply both sides, using left-multiplication, by

$$(E_{32,3}\, E_{31,-2}\, E_{21,1})^{-1} = E_{21,1}^{-1}\, E_{31,-2}^{-1}\, E_{32,3}^{-1}$$

$$= E_{21,-1}\, E_{31,2}\, E_{32,-3}$$

$(*)$ becomes

$$A = (\underbrace{E_{21,-1}\, E_{31,2}\, E_{32,-3}}_{L})\,\mathcal{U} = L\mathcal{U}$$

Note $E_{ij,\ell}$ is lower triangular if $i>j$.
(upper triangular if $i<j$)

ex. $E_{21,-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $E_{31,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ $E_{32,-3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

In particular, in our example with $A=LU$, L is a product of lower triangular matrices.

Fact: Product of lower triangular matrices is lower triangular.

(Similarly product of upper triangulars is upper triangular.)

We'll explain (prove) this fact below.

For now we get to the point: L is lower triangular.

So our example gives: $A = \underset{\text{lower triang.}}{L}\ \underset{\text{upper triang.}}{U}$

Let's compute L:

$$L = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21,-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_{31,2}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}}_{E_{32,-3}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

So we obtain:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{U}$$

There was something special about our example; there were no row exchanges in elimination.

Here is the general result:

Theorem Let A be an $n \times n$ matrix. Then A can be written as a product of matrices as follows:

$$A = PLU$$

P: permutation matrix
L: lower triangular
U: upper triangular

The permutation matrix P records the row exchanges.

Just as in our previous example, this theorem is explained by applying elimination to get A into Echelon Form.

Ex.

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow[\text{swap rows } 1 \,\&\, 3]{P_{13}} \begin{bmatrix} \boxed{-1} & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{E_{21,1}} \underbrace{\begin{bmatrix} \boxed{-1} & 1 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & \boxed{-1} \end{bmatrix}}_{\text{Echelon Form}} = U$$

So Elimination yields $E_{21,1} P_{13} A = U$

Multiply on left sides by $E_{21,-1} = E_{21,1}^{-1}$: we get

$$P_{13} A = E_{21,-1} U$$

Multiply on left sides by $P_{13} = P_{13}^{-1}$ to get

$$A = \underbrace{P_{13}}_{P} \underbrace{E_{21,-1}}_{L} U$$

Conclusion: $A = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$

Let's explain the "fact" we used, that

$$\underbrace{\begin{pmatrix} \text{lower} \\ \text{triangular} \end{pmatrix}}_{A} \times \underbrace{\begin{pmatrix} \text{lower} \\ \text{triangular} \end{pmatrix}}_{B} = \underbrace{\begin{matrix} \text{lower} \\ \text{triangular} \end{matrix}}_{AB}$$

Consider $n \times n$ matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$

a_{ij} with $j > i$

a_{ij} with $i = j$

a_{ij} with $i > j$

So A is lower triangular $\Leftrightarrow$ $a_{ij} = 0$ whenever $i < j$

B is lower triangular $\Leftrightarrow$ $b_{ij} = 0$ whenever $i < j$

To show AB is lower triangular
need to show every (i,j)-entry of AB, with $i<j$, is zero.

The (i,j)-entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \quad (\dagger)$$

Suppose $i<j$.
Consider a term $a_{ik}b_{kj}$ in expression $(\dagger)$.
If $i<k$ then $a_{ik}=0$ (since A lower triangular),
and then of course $a_{ik}b_{kj}=0$.
If on the other hand $i \geq k$ then

$$k \leq i < j \quad \text{so} \quad k<j.$$

Since B is lower triangular, $b_{kj}=0$, so $a_{ik}b_{kj}=0$.
We've shown that no matter what k is, each term $a_{ik}b_{kj}$ in $(\dagger)$ is zero. So $(\dagger)$ is zero, and we've shown that AB is lower triangular.