Elimination via matrices, Finding inverses   MTH 210

Recall from last time the following types of $n\times n$ matrices:

(1) diagonal matrices $D(a_1, a_2, \dots, a_n)$ ← multiplies row_i by a_i

(2) elimination matrices $E_{ij,\ell}$ ← adds $\ell \times (\text{row}_j)$ to row_i

(3) permutation matrices P_{ij} ← swaps rows i & j

As indicated, each of these matrix types corresponds to an elementary row operation.

We can recast the elimination algorithm using these matrices.

Ex:
$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{array}\right] \qquad P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(i) ↓ swap row_1 & row_2

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \qquad E_{32,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

(ii) ↓ add row_2 to row_3

$$D = D(1,1,\tfrac{1}{3}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{3} & 0 \end{array}\right] \xrightarrow[\text{(iii)}]{\text{multiply } \text{row}_3 \text{ by } 1/3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Echelon form

We can express the steps as follows

$$[\,A \mid \vec{b}\,] \xrightarrow{(i)} [\,P_{12}A \mid P_{12}\vec{b}\,] \xrightarrow{(ii)} [\,E_{32,1}P_{12}A \mid E_{32,1}P_{12}\vec{b}\,]$$

$$\big\downarrow (iii)$$

$$[\,DE_{32,1}P_{12}A \mid DE_{32,1}P_{12}\vec{b}\,]$$

at each step we multiply A & $\vec{b}$ by a matrix

Let's check this actually works!

$$DE_{32,1}P_{12}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1/3 \end{bmatrix}}_{DE_{32,1}}\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix}}_{P_{12}A}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

agrees w/ left side of Echelon form above!

Can similarly check the "$\vec{b}$" parts match up.

Elimination, recast: for $n \times n$ A Given $A\vec{x} = \vec{b}$, multiply both sides by some sequence of the 3 types of special matrices.

In the end we get $A'\vec{x} = \vec{b}'$ where A' is upper triangular.

In particular, $[A' | \vec{b}']$ is in Echelon or RREF.

This viewpoint can be very useful.

Let's go back to inverses.

Recall $\underset{n\times n}{A}$ is invertible if there's an $n\times n$ matrix A^{-1}

such that $AA^{-1} = \underset{\substack{n\times n \\ \text{identity}}}{I} = A^{-1}A$.

How can we find A^{-1}? Answer: Elimination!

(i) Write augmented matrix $[A \mid I]$

(ii) Do elimination algorithm to get A into RREF.

(iii) If A is invertible the end result will be

$$[I \mid A^{-1}]$$

In particular, the RREF of A is just I.

(iv) If the RREF of A is not I, A is not invertible.

Ex.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$. Find A^{-1}.

elimination:
(gives the "what")

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 & 1 & 0 \\ 0 & 0 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 & 1 & 0 \\ 0 & 0 & \boxed{1} & \tfrac{1}{2} & 0 & -\tfrac{1}{2} \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & \tfrac{1}{2} & 0 & \tfrac{1}{2} \\ 0 & \boxed{1} & 0 & 0 & 1 & 1 \\ 0 & 0 & \boxed{1} & \tfrac{1}{2} & 0 & -\tfrac{1}{2} \end{array}\right]$$

RREF of A — this is A^{-1}

matrix mult. version:
(gives the "why")

$$\left[\, A \mid I \,\right]$$

$$\downarrow$$

$$\left[\, E_{31,-1} E_{21,1} A \mid E_{31,-1} E_{21,1} \,\right]$$

$$\downarrow \quad D = D(1, 1, -\tfrac{1}{2})$$

$$\left[\, D E_{31,-1} E_{21,1} A \mid D E_{31,-1} E_{21,1} \,\right]$$

$$\downarrow$$

$$\left[\, \underbrace{E_{13,-1} E_{12,-2} D E_{31,-1} E_{21,1}}_{B} A \;\Big|\; \underbrace{E_{13,-1} E_{12,-2} D E_{31,-1} E_{21,1}}_{B} \,\right]$$

$$\parallel$$

$$\left[\, BA \mid B \,\right]$$

We see why this works:

elimination gives $[BA \mid B]$ where B = product of special matrices and BA = RREF of A. So if the RREF of A is I,

then $BA = I$, meaning... $B = A^{-1}$!

It's always good to double check we found A^{-1} correctly.

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \qquad AA^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex. Determine if $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ is invertible.

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow$$

RREF of A is not I, so A is not invertible.

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{array}\right]$$

RREF of A

Here is why, if RREF of A is not I, then A is not invertible.

First look at all RREF's of 2×2 matrices:

$$I = \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & ? \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Possibilities for the 3×3 case:

$$I = \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & ? & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & 0 & ? \\ 0 & \boxed{1} & ? \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \boxed{1} & ? & ? \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & \boxed{1} & ? \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We notice: if RREF of A is not I then it has a row of all zeros.

Then you can find a $\vec{b}$ in $\mathbb{R}^n$ such that $[A \mid \vec{b}]$ after elimination has a row of the form

$$[0 \quad 0 \cdots 0 \mid c], \qquad c \neq 0$$

and thus $A\vec{x} = \vec{b}$ has no solutions.

But if A is invertible, $A\vec{x} = b$ always has solution $\vec{x} = A^{-1}\vec{b}$.

So indeed, if RREF of A is not I, A is not invertible!

<u>Ex</u>.

Determine if $A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ is invertible (if so, compute A^{-1}).

$$\left[\begin{array}{ccc|ccc} 0 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & 0 & 1 \\ 0 & 0 & \boxed{-1} & 1 & 0 & 0 \end{array}\right]$$

$row_1 \to row_3$
$row_2 \to row_1$
$row_3 \to row_1$

Scale rows 2 & 3 by -1 $\longrightarrow$
$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 & 0 & 0 \end{array}\right]$$

So we get $A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$.

As usual it's good to double check that $AA^{-1} = I$ (or $A^{-1}A = I$).

Through our method, here is another characterization:

$\underset{n \times n}{A}$ is invertible $\iff$ A has n pivots in the elimination algorithm

(the maximal possible # of pivots!)

Here's another criterion:

$\underset{n \times n}{A}$ invertible $\iff$ the only solution $\vec{x}$ to $A\vec{x} = \vec{0}$ is the zero vector: $\vec{x} = \vec{0}$