Inverse Matrices

MTH210 2/16/23

A: $n\times n$ matrix (square matrix)

an $n\times n$ matrix B is an inverse of A if

$$\underset{\substack{\\ n\times n \text{ identity}}}{AB = I}$$

and $BA = I$.

If A has an inverse, the inverse is unique (there is only one). To see this, suppose B, C are both inverses for A. This means B, C are $n\times n$ and:

$$BA = I \underset{(*)}{=} AB \qquad CA \underset{(**)}{=} I = AC$$

We then compute:

$$B = IB \overset{(**)}{=} (CA)B = C(AB) \overset{(*)}{=} CI = C.$$

So $B=C$. Thus any two inverses are equal, as claimed.

Therefore we can speak of the inverse of A (if it exists!).

If A has an inverse we write A^{-1} for it.

If A has an inverse we also say A is invertible.

So if A is invertible we have a unique A^{-1} satisfying:

$$\boxed{AA^{-1} = \underset{n\times n \text{ identity}}{I} = A^{-1}A}$$

Note: to show a matrix A^{-1} is in fact the inverse of A, you only need to check one of $AA^{-1} = I$ or $I = A^{-1}A$.

Inverses are very useful! Suppose we want to solve

$$A\vec{x} = \vec{b}$$

where $\underset{n\times n}{A}$ and $\vec{b}$ in $\mathbb{R}^n$ are given, with $\vec{x}$ in $\mathbb{R}^n$ the unknown. If A is invertible, multiply both sides (using left multiplication) by A^{-1}:

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

$$(A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

Conclusion:

$$\vec{x} = A^{-1}\vec{b}$$

If A is an $n\times n$ invertible matrix then there is a unique solution to

$$A\vec{x} = \vec{b}$$

for any given $\vec{b}$ in $\mathbb{R}^n$. The solution is $\vec{x} = A^{-1}\vec{b}$.

Ex.

1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ claim: $A^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$

2x2

To verify claim, compute:

$$AA^{-1} = \begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} -2 \\ 3/2 \end{matrix}} & \begin{matrix} 1 \\ -1/2 \end{matrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underset{\text{2x2 identity}}{I}$$

2) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ claim: A has no inverse

One way to verify:

Do elimination for $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} \boxed{1} & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightsquigarrow x + 2y = 0, \text{ a line of solutions.}$$

So there is not a unique (one) solution.
We saw above that if A were invertible, there'd be <u>one</u> solution.
So A is <u>not</u> invertible.

3) Here's a formula for A^{-1} in the 2×2 case:

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\boxed{A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$

ad-bc is called the <u>determinant</u> of A.

In particular, for A^{-1} to exist we need

$$ad - bc \neq 0 \quad \text{(non-zero det)}$$

In previous example,

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \begin{matrix} a=1 & b=2 \\ c=2 & d=4 \end{matrix} \qquad \text{so} \quad ad-bc = (1)(4)-(2)(2) = 0.$$

So again we see a reason for why A is not invertible.

Inverses of products:

If A, B are invertible $n \times n$ matrices then

AB is invertible and

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

let's prove this. Need to check $(B^{-1}A^{-1})(AB) = I$

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB$$

$$= B^{-1}B = I. \quad \text{Done.}$$

Iterating this identity, we get:

if $A_1, A_2, \ldots, A_K$ are invertible $n \times n$ matrices then $\boxed{(A_1A_2 \cdots A_K)^{-1} = A_K^{-1}A_{K-1}^{-1} \cdots A_1^{-1}}$

note the order reversal!

<u>Ex</u>. $n \times n$ <u>Elimination matrix</u> $E_{ij,\ell}$ $(i \neq j)$

$$\boxed{E_{ij,\ell} = \begin{array}{l} n\times n \text{ identity matrix with one extra} \\ \text{entry: the } (i,j)\text{-entry is } \ell. \end{array}}$$

3×3 case: $E_{31,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

(3,1)-entry

$$E_{31,2}\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z + 2x \end{bmatrix}$$

one of our elementary row operations!

So $E_{31,2}$ has effect of "adding $2 \times (\text{row}_1)$ to row_3".

Generally: $E_{ij,\ell}$ "adds $\ell \times (\text{row}_j)$ to row_i".

Inverse of $E_{31,2}$ should be "add $(-2) \times (\text{row}_1)$ to row_3"

which is $E_{31,-2}$. And it is!

$$E_{31,2}\,E_{31,-2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Thus $E_{31,2}^{-1} = E_{31,-2}$. In general, $E_{ij,\ell}^{-1} = E_{ij,-\ell}$.

Ex $n\times n$ Permutation matrix P_{ij} is the identity $\overset{n\times n}{I}$ but w/ the i^{th} and j^{th} rows swapped.

3×3 case: $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$P_{12}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

P_{12} acts on vectors by swapping 1^{st} & 2^{nd} coordinates

(corresponds to "row exchange" operation)

Note $P_{ij} = P_{ji}$.

Swapping rows i & j twice has no effect.

Interpretation: $P_{ij}^2 = P_{ij}P_{ij} = I$. (so $P_{ij}^{-1} = P_{ij}$!)

ex:

$$P_{12}P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

More generally, a permutation matrix is any product of P_{ij}'s.

ex. $P_{12}P_{23} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ($\neq P_{ij}$ for any i,j !)

Ex. Diagonal matrices

$D(a_1, a_2, \dots, a_n)$ = diagonal matrix w/ entries $a_1, \dots, a_n$ along the diagonal

$n \times n$

$$= \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & a_n \end{bmatrix}$$

ex.

$$D(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad D(1,2,3)\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$$

($D(a_1, \dots, a_n)$ scales row_i by a_i)

Note $D(1, \frac{1}{2}, \frac{1}{3})D(1,2,3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = I$

In general:

diagonal $n \times n$ $D(a_1, \dots, a_n)$ is invertible $\Leftrightarrow$ $a_1, \dots, a_n$ all non-zero.

and in this case,

$$\boxed{D(a_1, a_2, \dots, a_n)^{-1} = D\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)}$$