Matrix Multiplication & other Operations

Before starting this new topic let's review the elimination algorithm.

Ex.

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \end{array}\right] \xrightarrow{\text{row swap}} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right]$$

$\downarrow$ add row 2 to row 3

$$\text{REF} \quad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] \xleftarrow{\text{mult. row 3 by } 1/3} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 0 & 0 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & \boxed{3} & 0 \end{array}\right]$$

Echelon form

subtract $2\times(\text{row }3)$ from row 2 $\downarrow$ then subtract row 2 from row 1

$$\text{RREF} \quad \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] \rightsquigarrow \begin{cases} x = -1 \\ y = 1 \\ z = 0 \end{cases} \quad \text{done!}$$

Geometry?

3 planes in $\mathbb{R}^3$ intersect at one point.

(In going from REF to RREF, we should always go "bottom-to-top": use the lowest pivot to get zeros above it; then the next lowest...)

Now onto matrix operations. Suppose we have:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & & \vdots \\ \vdots & & \ddots & \\ b_{n1} & \cdots & & b_{nk} \end{bmatrix}$$

$m\times n$ $\qquad$ $n\times k$

$$\underset{m\times n \text{ matrix}}{A}, \quad \underset{n\times k \text{ matrix}}{B} \xrightarrow[\text{multiplication}]{\text{matrix}} \text{"}AB\text{"} \text{ new matrix of size } m\times k.$$

The (i,j)-entry (row i, column j) of AB is defined to be

$$(\text{row}_i \text{ of } A)\cdot(\text{col}_j \text{ of } B)$$

$$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Thus we have:

$$AB = \begin{bmatrix} \text{row}_1^A\cdot\text{col}_1^B & \text{row}_1^A\cdot\text{col}_2^B & \cdots & \text{row}_1^A\cdot\text{col}_k^B \\ \text{row}_2^A\cdot\text{col}_1^B & \text{row}_2^A\cdot\text{col}_2^B & & \vdots \\ \vdots & & \ddots & \\ \text{row}_m^A\cdot\text{col}_1^B & \cdots & \cdots & \text{row}_m^A\cdot\text{col}_k^B \end{bmatrix}$$

Size of AB is easy to find:

$$\left(\underset{m\times \cancel{n}}{A}\right)\left(\underset{\cancel{n}\times k}{B}\right) = \underset{m\times k}{AB}$$

Examples.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

2×3 $\qquad$ 3×2

AB will be 2×2.

$$AB = \begin{bmatrix} \boxed{2 \quad 0 \quad 1} \\ \boxed{-1 \quad 3 \quad 0} \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix}-1\\1\\1\end{matrix}} & \boxed{\begin{matrix}1\\-1\\0\end{matrix}} \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 4 & -4 \end{bmatrix}$$

What about BA?

$$\underset{3\times\cancel{2}}{(B)}\,\underset{\cancel{2}\times 3}{(A)} = \underset{3\times 3}{BA}$$

$$BA = \begin{bmatrix} \boxed{-1 \quad 1} \\ \boxed{1 \quad -1} \\ \boxed{1 \quad 0} \end{bmatrix} \left[\boxed{\begin{matrix} 2 \\ -1 \end{matrix}} \; \boxed{\begin{matrix} 0 \\ 3 \end{matrix}} \; \boxed{\begin{matrix} 1 \\ 0 \end{matrix}} \right] = \begin{bmatrix} -3 & 3 & -1 \\ 3 & -3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

Another viewpoint:

Think of $\underset{n\times k}{B}$ in terms of columns: $B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_k \\ | & | & & | \end{bmatrix}$

Then given $m\times n$ matrix A we have:

$$\underset{m\times k}{AB} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_k \\ | & | & & | \end{bmatrix}$$

In this way, matrix multiplication is a very natural extension of matrices acting on vectors.

(There's also a complementary viewpoint where A is decomposed into rows.)

<u>Ex</u>

$A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$ $2\times\cancel{2}$ $\quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $\cancel{2}\times 3$ $\quad$ AB will be 2×3.

$$AB = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \left[\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right]$$

$$= \left[\begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right] = \begin{bmatrix} 2 & 5 & 2 \\ 4 & 3 & 4 \end{bmatrix}.$$

Yet another viewpoint:

Write A as columns, B as rows

$$A = \begin{bmatrix} | & & | \\ col_1^A & \cdots & col_n^A \\ | & & | \end{bmatrix} \qquad B = \begin{bmatrix} — & row_1^B & — \\ & \vdots & \\ — & row_n^B & — \end{bmatrix}$$

Then $$AB = \underbrace{\underset{m\times 1}{col_1^A}\ \underset{1\times k}{row_1^B}}_{m\times k} + \underbrace{\underset{m\times 1}{col_2^A}\ \underset{1\times k}{row_2^B}}_{m\times k} + \cdots \underbrace{\underset{m\times 1}{col_n^A}\ \underset{1\times k}{row_n^B}}_{m\times k}$$

This represents AB as a sum of $m\times k$ matrices.

Ex. $A = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix}$

$$AB = col_1^A\, row_1^B + col_2^A\, row_2^B = \begin{bmatrix} \boxed{1} \\ -1 \end{bmatrix} \begin{bmatrix} \boxed{0} & \boxed{2} \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{0} & \boxed{2} \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & -2 \end{bmatrix}.$$

Special case:

$\vec{v}$ vector in $\mathbb{R}^n$ can be viewed as a $n\times 1$ matrix.

Then if A is $m\times n$, $\underset{m\times \cancel{n}\ \cancel{n}\times 1}{A\vec{v}}$ is $m\times 1$, i.e. a vector in $\mathbb{R}^m$.

(Agrees w/ previous def. of $A\vec{v}$!)

Two other basic operations:

Addition: $\underset{m\times n}{A} + \underset{m\times n}{B}$ is an $m\times n$ matrix

same size!

Just add componentwise!

(we actually already used this operation in the previous example.)

Scalar multiplication: $\underset{m\times n}{A}$ $\quad \underset{\text{scalar}}{c}$ $\rightsquigarrow$ $\underset{m\times n}{cA}$ each entry of A is multiplied by c.

Ex. $5\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -10 & 5 \end{bmatrix}$ $\quad -2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$.

Some **Rules**: (you can check all of these!)

A, B, C matrices
d scalar

$$A + B = B + A$$
$$d(A+B) = dA + dB$$
$$A + (B+C) = (A+B) + C$$
$$(A+B)C = AC + BC$$
$$A(B+C) = AB + AC$$
$$(AB)C = A(BC)$$

Warning:

In general, $AB \neq BA$!

Ex. $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

So we see $AB \neq BA$.

Also, if $AB = 0$ (the zero matrix),

it is not necessarily the case that one of A, B is zero.

Ex.

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = B$. Then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. zero matrix

So AB is the 2×2 zero matrix, but A and B are not zero matrices.

Ex. Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Compute A^n for any $n = 1, 2, 3, \ldots$

$$A^2 = AA = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix}$$

$$A^4 = AA^3 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 8 & 7 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 15 \\ 0 & 1 \end{bmatrix} \ldots$$

see the pattern?

$$A^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$$