Elimination (Intro)

Our newest formulation of Linear Algebra: solve

$$\boxed{A\vec{x} = \vec{b}} \quad (*)$$

where A is an $m \times n$-matrix, $\vec{b}$ vector in $\mathbb{R}^m$.
Here $\vec{x}$ is the "unknown", a vector in $\mathbb{R}^n$.

Recall $(*)$ is equivalent to:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \ddots \qquad\quad \vdots &\quad\ \ \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

First suppose system is "upper triangular", i.e. all constants a_{ij} below staircase are zero. For simplicity assume $\underline{m=n}$.

(# equations = # unknowns)

So we're looking at:

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\ddots \qquad \vdots \quad &\quad \vdots \\
a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n &= b_{n-1} \\
a_{nn}x_n &= b_n
\end{aligned}$$

"Upper Triangular System"

Let's solve this. First solve for x_n using last eq.:

$$x_n = b_n / a_{nn}$$

then substitute x_n into 2^{nd} to last eq:

$$a_{n-1,n-1} x_{n-1} + a_{n-1,n}\left(\frac{b_n}{a_{nn}}\right) = b_{n-1}$$

and we can solve for x_{n-1}.

Then plug x_{n-1}, x_n into 3^{rd} to last eq.; can solve for x_{n-2}.

We keep going and eventually solve for all $x_1, x_2, \ldots, x_n$.

Ex. $\begin{cases} x+y-z=0 \\ 2y+z=1 \\ 3z=-1 \end{cases}$ note this is equivalent to:

$$\overset{A}{\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}} \overset{\vec{x}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$$

"upper triangular" matrix

From last eq, $z = -\frac{1}{3}$.

Then $2y + z = 1$ becomes $2y + (-\frac{1}{3}) = 1 \rightarrow 2y = \frac{4}{3} \rightarrow y = 2/3$

Substitute $y = 2/3$, $z = -1/3$ into $x+y-z=0$ to get

$$x + \frac{2}{3} - \left(-\frac{1}{3}\right) = 0$$

$$\rightarrow x + 1 = 0 \rightarrow x = -1.$$

Solution is $\vec{x} = \begin{bmatrix} -1 \\ 2/3 \\ -1/3 \end{bmatrix}$.

Takeaway: We know how to solve upper triangular systems. They're "easy".

Idea of Elimination:

Given any system of linear eqs., transform it into an "equivalent" system which is upper triangular.

("equivalent" means the systems have same solutions — not that they have the same equations!)

Geometric intuition:

these (2×2) systems have same solution (intersection point p) even though equations (the lines) are not same.)

Let's illustrate this idea of elimination by example.

Ex.
$$\begin{cases} x + y - z = 1 & (1) \\ 2x - y + z = -1 & (2) \\ -x + 2y + 2z = 0 & (3) \end{cases}$$

Choose a row w/ a nonzero coeff. in front of x. The coefficient of in this row is called a *pivot*.

We use this pivot to eliminate x from equations below.

$$\begin{cases} \boxed{1}x + y - z = 1 \\ 2x - y + z = -1 \\ -x + 2y + 2z = 0 \end{cases}$$

1st pivot

For example, (2) − 2(1) eliminates x from second row.

$$\begin{cases} \boxed{1}x + y - z = 1 & ① \\ -3y + 3z = -3 & ②-2① \\ 3y + z = 1 & ③+1① \end{cases}$$

Also ③ + 1 × ① eliminates x from 3rd eq.

We next choose a pivot for the "y" column.

$$\begin{cases} \boxed{1}x + y - z = 1 \\ \boxed{-3}y + 3z = -3 \\ 3y + z = 1 \end{cases}$$

Add second row to third, using the pivot to eliminate y in 3rd eq.

$$\rightarrow \begin{cases} \boxed{1}x + y - z = 1 \\ \boxed{-3}y + 3z = -3 \\ \boxed{4}z = -2 \end{cases}$$

We now have an upper triangular system!

Solve it! $z = -2/4 = -1/2$ from ③

② becomes $-3y + 3(-\frac{1}{2}) = -3 \rightarrow -3y = -3/2 \rightarrow y = \frac{1}{2}$

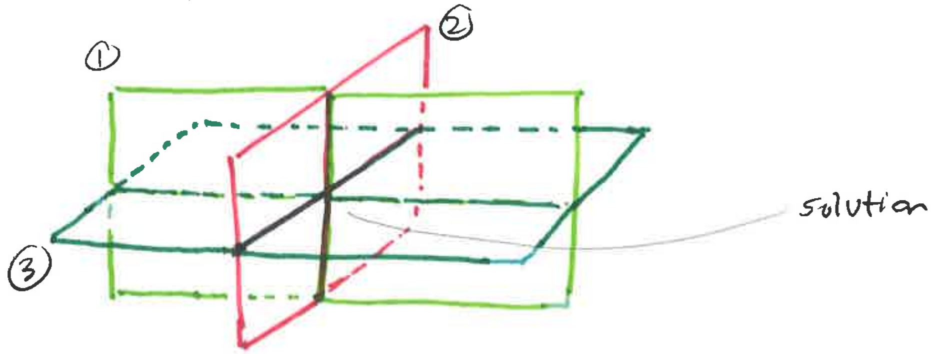
① becomes $x + (\frac{1}{2}) - (-\frac{1}{2}) = 1 \rightarrow x + 1 = 1 \rightarrow x = 0.$

Solution: $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}.$

Geometric Interpretation:

①, ②, ③ each define a plane in $\mathbb{R}^3$

Three planes typically intersect in one point, That's what happens here (our solution is the intersection). ⑤



In each step of our solution we're replacing one plane with some new plane (ex. replacing ② by ② - 2·①) without changing the intersection point.

So: picture actually changes at each step!
But the solution does not!

We do elimination faster if we use matrices.

Ex. Consider

$$\begin{cases} x + 2y - z = 1 \\ 2x - y - z = 3 \\ 3x + y + z = 0 \end{cases}$$

this is the same as $A\vec{x} = \vec{b}$

where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

We form the "Augmented Matrix":

$$[A \mid \vec{b}\,] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 1 & 0 \end{array}\right]$$

Now proceed as before:

Old way | New way

pivot

$$\begin{cases} \boxed{1}x + 2y - z = 1 & (1) \\ 2x - y - z = 3 & (2) \\ 3x + y + z = 0 & (3) \end{cases} \qquad \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 2 & -1 & -1 & 3 \\ 3 & 1 & 1 & 0 \end{array}\right]$$

$\downarrow$

$$\begin{cases} \boxed{1}x + 2y - z = 1 & (1) \\ -5y + z = 1 & (2) - 2\times(1) \\ -5y + 4z = -3 & (3) - 3\times(1) \end{cases} \qquad \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 0 & -5 & 1 & 1 \\ 0 & -5 & 4 & -3 \end{array}\right]$$

$\downarrow$

new pivot

$$\begin{cases} \boxed{1}x + 2y - z = 1 \\ \boxed{-5}y + z = 1 \\ 3z = -4 & \text{(subtract 2nd eq. from 3rd)} \end{cases} \qquad \left[\begin{array}{ccc|c} \boxed{1} & 2 & -1 & 1 \\ 0 & \boxed{-5} & 1 & 1 \\ 0 & 0 & 3 & -4 \end{array}\right]$$

We're left with an upper triangular system. Solve!

$3z = -4 \rightarrow z = -4/3$

$-5y + z = 1 \rightarrow -5y + (-4/3) = 1 \rightarrow -5y = \frac{7}{3} \rightarrow y = -\frac{7}{15}$

$x + 2y - z = 1 \rightarrow x + 2\left(\frac{-7}{15}\right) - \left(\frac{-4}{3}\right) = 1 \rightarrow x + \frac{6}{15} = 1 \rightarrow x = \frac{9}{15} = \frac{3}{5}$

Solution: $\vec{x} = \begin{bmatrix} 3/5 \\ -7/15 \\ -4/3 \end{bmatrix}$.

Basic ingredients of elimination:

the 3 elementary row operations

(each "row" corresponds to an equation)

(1) multiply a row by a non-zero scalar

(2) add/subtract a multiple of one row to another

(3) row exchange: swap two rows

Key reason why elimination works:

all the elementary row operations do not change the set of solutions to the system of eqs.