Matrices

$m \times n$ matrix is an array of #'s w/ m rows, n columns

$$A = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}} \Bigg\} m \text{ rows}$$

$m \times n$ matrix A "acts on" length n vectors $\vec{x}$ in $\mathbb{R}^n$:

we define

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}}_{\substack{A \\ m \times n}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} \overset{\text{def}}{=} \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Observe

A $m \times n$ matrix, $\vec{x}$ length n vector $\rightarrow$ $A\vec{x}$ is a vector in $\mathbb{R}^m$, length m

Why are we talking about matrices?

Recall: system of m linear equations in n unknowns:

$$(*)\ \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \quad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

$x_1, \dots, x_n$ variables

rest are constants

We see that $(*)$ is equivalent to:

$$\boxed{A\vec{x} = \vec{b}}$$

where $\underset{m\times n}{A}$ matrix of a_{ij} terms

and $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ is variable, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ fixed.

Studying matrices will help us with solving linear systems.

Way to think about $A\vec{x}$:

$$A\vec{x} = \underbrace{\begin{array}{c} \text{row}_1 \\ \begin{bmatrix} \boxed{a_{11} \ a_{12} \ \cdots \ a_{1n}} \\ a_{21} \ a_{22} \quad \vdots \\ \vdots \quad \ddots \quad \vdots \\ \boxed{a_{m1} \ a_{m2} \ \cdots \ a_{mn}} \end{bmatrix} \\ \text{row}_m \end{array}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \text{row}_1 \cdot \vec{x} \\ \text{row}_2 \cdot \vec{x} \\ \vdots \\ \text{row}_m \cdot \vec{x} \end{bmatrix}$$

Ex.

1) $\underset{2\times 2}{A} = \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} \boxed{1 \quad 2} \\ \boxed{-1 \quad 0} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 2 \end{matrix}} \end{bmatrix} = \begin{bmatrix} (1)(3) + (2)(2) \\ (-1)(3) + (0)(2) \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

2) $\underset{3\times 2}{A} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \\ 2 & 1 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} \boxed{5 \quad 0} \\ \boxed{0 \quad -1} \\ \boxed{2 \quad 1} \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \begin{bmatrix} (5)(1) + (0)(1) \\ (0)(1) + (-1)(1) \\ (2)(1) + (1)(1) \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3) $\underset{2\times 3}{A} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

4) $\underset{3\times 3}{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

5) $\underset{1\times 3}{A} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ then $A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \end{bmatrix} = -\frac{5}{2}$

$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -1 \end{bmatrix}$

Let's look at how to write eqs. using matrices.

Ex.

$$\begin{cases} x + y + z = 0 \\ x - y + 2z = 1 \\ 2x + z = 2 \end{cases}$$

is equivalent to $A\vec{x} = \vec{b}$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

We can also think of this system using "column picture":

$$x\begin{bmatrix}1\\1\\2\end{bmatrix} + y\begin{bmatrix}1\\-1\\0\end{bmatrix} + z\begin{bmatrix}1\\2\\1\end{bmatrix} = \begin{bmatrix}0\\1\\2\end{bmatrix}$$ (vector equation)

How is this related to matrices?

Column picture of $A\vec{x}$:

$$\underbrace{\begin{bmatrix} | & | & & | \\ \text{Col}_1 & \text{Col}_2 & \cdots & \text{Col}_n \\ | & | & & | \end{bmatrix}}_{A\ m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\vec{x}} = x_1\begin{bmatrix} | \\ \text{Col}_1 \\ | \end{bmatrix} + x_2\begin{bmatrix} | \\ \text{Col}_2 \\ | \end{bmatrix} + \cdots + x_n\begin{bmatrix} | \\ \text{Col}_n \\ | \end{bmatrix}$$

Ex.

1) $A = \begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}$

$\vec{x} = \begin{bmatrix}3\\2\end{bmatrix}$

$$A\vec{x} = \begin{bmatrix}1 & 2\\-1 & 0\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = 3\begin{bmatrix}1\\-1\end{bmatrix} + 2\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}3+4\\-3+0\end{bmatrix} = \begin{bmatrix}7\\-3\end{bmatrix}$$

2) $A = \begin{bmatrix}5 & 0\\0 & -1\\2 & 1\end{bmatrix}$

$\vec{x} = \begin{bmatrix}1\\1\end{bmatrix}$

$$A\vec{x} = 1\begin{bmatrix}5\\0\\2\end{bmatrix} + 1\begin{bmatrix}0\\-1\\1\end{bmatrix} = \begin{bmatrix}5\\-1\\3\end{bmatrix}$$

Some important matrices:

Identity matrix: $n \times n$

$I_{1\times1} = [1]$ $\quad I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$I = I_{n \times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

(drop n if clear from context)

always a square matrix! (# cols = # rows)

If $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ then $I\vec{x} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & \cdots & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \vec{x}.$$

Thus $\boxed{I\vec{x} = \vec{x}.}$

Zero matrix $m \times n$

$O_{m\times n}$ = $m \times n$ matrix w/ all entries zero.

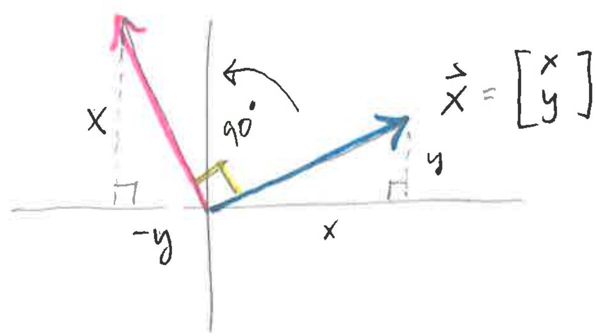
(sometimes just write $O = O_{m\times n}$)

ex $O_{2\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Key property: $\boxed{O_{m\times n}\vec{x} = \vec{0}}$

* Matrices act on vectors. They do things! *

Ex. Find a 2×2 matrix A that acts as 90° rotation (counter clockwise).



If $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ we want $A\vec{x} = \begin{bmatrix} -y \\ x \end{bmatrix}$.

(this is 90° rot of $\vec{x}$)

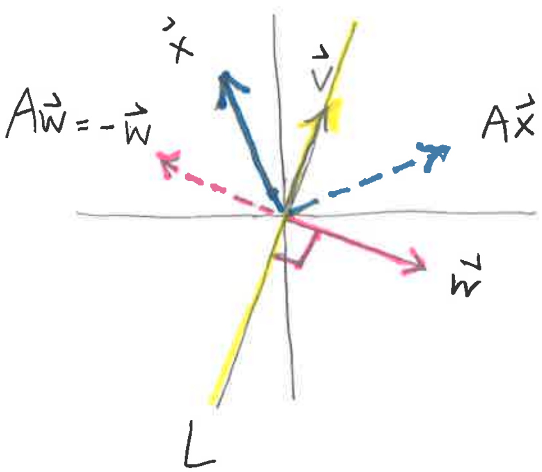
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $A\vec{x} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} -y \\ x \end{bmatrix}$ gives $\begin{cases} ax+by = -y \\ cx+dy = x \end{cases}$ $a = d = 0$, $b = -1$, $c = 1$ works.

So $\boxed{A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}$ is 90° counterclockwise rotation.

<u>Ex</u>. line $L = \{ t\vec{v} \mid t \in \mathbb{R} \}$. $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Find a 2x2 matrix A so that $A\vec{x}$ is reflection of $\vec{x}$ across the line L.



$\vec{v}$ reflected across L is just $\vec{v}$

So we should have $\boxed{A\vec{v} = \vec{v}}$ (1)

Let $\vec{w}$ be $\perp$ to $\vec{v}$.

We can take $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

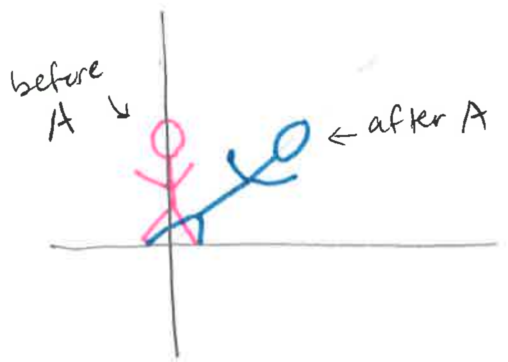
Then we should have $\boxed{A\vec{w} = -\vec{w}.}$ (2)

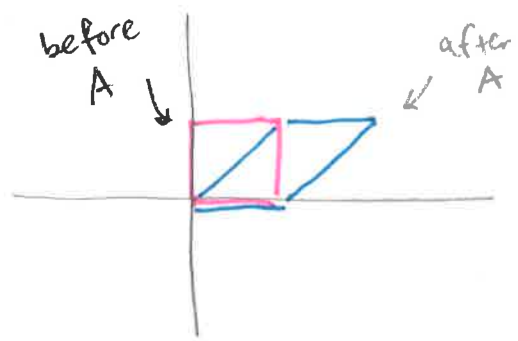
(1) gives $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{cases} a+2b = 1 \\ c+2d = 2 \end{cases}$

(2) gives $\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\vec{w}} = \underbrace{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}_{-\vec{w}} \to \begin{cases} 2a-b = -2 \\ 2c-d = 1 \end{cases}$

Solve to get $a = -3/5$ $b = 4/5$ $c = 4/5$ $d = 3/5$ $\longrightarrow$ $\boxed{A = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}}$

Ex. ("Shear map")





Determine a 2×2 matrix A that transforms pictures in $\mathbb{R}^2$ by shearing them in the horizontal direction, as depicted.

Specifically, A should fix the x-axis, and it should send the y-axis to the line $y=x$. Also sends horizontal lines to horizontal lines.

Conditions: (1) $A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$ (since A fixes x-axis)

(2) $A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$ (see 2nd picture above)

(1) $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \to \begin{cases}a=1\\c=0\end{cases}$

(1) $\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \to \begin{cases}b=1\\d=1\end{cases}$

$\to \boxed{A=\begin{bmatrix}1&1\\0&1\end{bmatrix}}$