[Cont. of Planes in $\mathbb{R}^3$] 1/26/23

Example (in $\mathbb{R}^3$)

Write the plane P given by $x - 2y - z = 1$

in parametric form $P = \{\vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R}\}$.

One way: find some points on P

then use (point, point, point) method

Quicker way: Let $y = s$, $z = t$.

Then $x - 2y - z = 1 \to x - 2s - t = 1$

$\to x = 1 + 2s + t$

Then our variable point $\vec{x}$ on plane P is

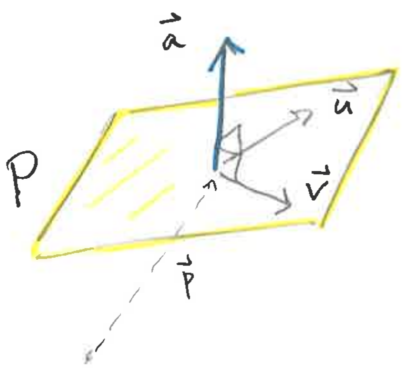
$$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 + 2s + t \\ s \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{p}} + s\underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + t\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

Thus $P = \{\vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R}\}$

with $\vec{p}, \vec{u}, \vec{v}$ as above.

Given a plane P of the form $P = \{\vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R}\}$

how can we write it as $\vec{a} \cdot \vec{x} = d$?



Given vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ is there a way to find a vector $\vec{a}$ perpendicular to $\vec{u}, \vec{v}$?

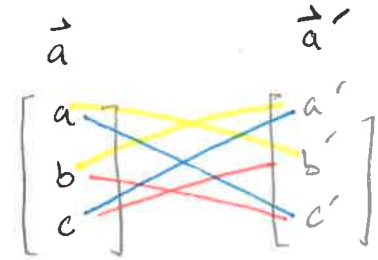
Yes. Here's how.

Cross Product (this is specific to $\mathbb{R}^3$!)

Given $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\vec{a}' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix}$ the cross product is

$$\vec{a} \times \vec{a}' = \begin{bmatrix} bc' - b'c \\ a'c - ac' \\ ab' - a'b \end{bmatrix}$$

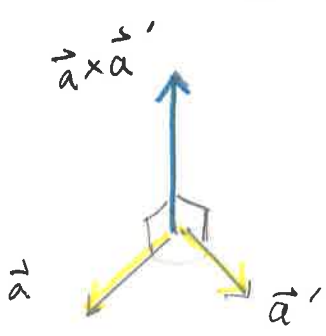
← note signs here



note: (vector) x (vector) = vector

Key Property:

$\vec{a} \times \vec{a}'$ is $\perp$ to both $\vec{a}$ & $\vec{a}'$



order matters! $\vec{a} \times \vec{a}' = -\vec{a}' \times \vec{a}$

← direction of $\vec{a} \times \vec{a}'$ determined by "right hand rule."

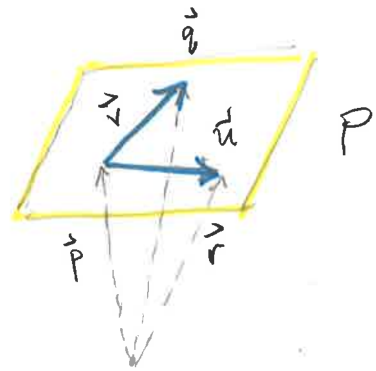
Example Let a plane P in $\mathbb{R}^3$ be determined by the following 3 points on the plane:

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \qquad \vec{q} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{r} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

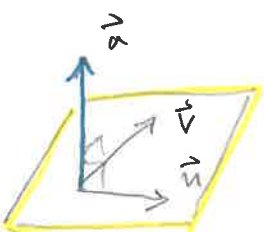
Write P in the form $\vec{a} \cdot \vec{x} = d$.

Direction vectors for P:

$$\vec{u} = \vec{r} - \vec{p} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix}$$



$$\vec{v} = \vec{q} - \vec{p} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$



To get $\vec{a}$ which is $\perp$ to $\vec{u}, \vec{v}$ we let $\vec{a} = \vec{u} \times \vec{v}$:

$$\vec{u} \times \vec{v} = \begin{bmatrix} -1 \\ -3 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} (-3)(-1) - (-1)(-1) \\ (-1)(+1) - (-1)(-1) \\ (-1)(-1) - (-3)(1) \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \overset{\text{def}}{=} \vec{a}$$

Know $\vec{x} \cdot \vec{a} = d = \vec{a} \cdot \vec{p}$

$$\vec{a} \cdot \vec{p} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = (2)(1) + \cancel{(-2)(0)} + (4)(2) = 2 + 8 = 10$$

Thus $\vec{a} \cdot \vec{x} = d$ where $\vec{a} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$, $d = 10$.

Written out, this is

$$\underbrace{2x - 2y + 4z}_{\vec{a}\cdot\vec{x}} = \underbrace{10}_{d}.$$

(An Aside: the cross product "$\times$" is only defined on $\mathbb{R}^3$.

For which $\mathbb{R}^n$ is there a (non-zero) "product" that takes vectors $\vec{u}, \vec{v}$ and gives a vector $\perp$ to $\vec{u}, \vec{v}$?

$$\left.\begin{array}{l} \mathbb{R}^3 \leftarrow \text{cross product} \\ \mathbb{R}^7 \leftarrow \text{"7d cross product"} \end{array}\right\} \text{only possibilities.}$$)

Vectors & Linear Equations

Described lines, planes, ... what's next?

Understand their intersections.

"Geometric interpretation" of Linear Algebra:

study of intersections of hyperplanes

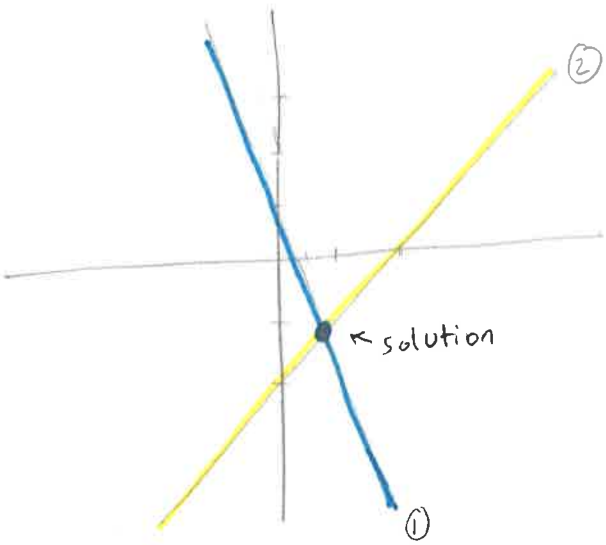
<u>Example</u> Find (x,y) in $\mathbb{R}^2$ satisfying

$$\begin{cases} 3x+y=1 & (1) \\ x-y=2 & (2) \end{cases}$$

system of 2 linear eqs.

(1), (2) each describe a line in $\mathbb{R}^2$.

Solutions of (1) & (2) are intersection points of the lines



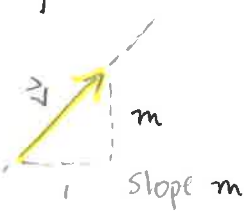
Geometry tells us there's a <u>unique</u> solution.

Some ways to find the soln:

1) Express (2) in parametric form

$$L = \{\vec{p} + t\vec{v} \mid t \in \mathbb{R}\}$$

$\vec{p} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



(line of slope m has direction vector $\begin{bmatrix} 1 \\ m \end{bmatrix}$)

Thus line (2) is

$$L = \left\{ \begin{bmatrix} 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

So an arbitrary point on line (2) is of the form:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t-2 \end{bmatrix}$$

Then plug this into ①: $3x+y=1$

$$3(t)+(t-2)=1 \rightarrow 4t-2=1 \rightarrow t=3/4$$

So $\vec{x}=\begin{bmatrix} x \\ y \end{bmatrix} \underset{\uparrow\, t=3/4}{=} \begin{bmatrix} 3/4 \\ \frac{3}{4}-2 \end{bmatrix} = \begin{bmatrix} 3/4 \\ -5/4 \end{bmatrix}$ is the solution.

another method

2) Elimination $\begin{cases} 3x+y=1 & ① \\ x-y=2 & ② \end{cases}$

Combine ① & ② to eliminate x or y.

③ = ② + ①:

$$\begin{array}{rll} x-\cancel{y}=2 & ② & \\ +\ (3x+\cancel{y}=1) & ① & \rightarrow \quad x=3/4 \\ \hline 4x\ =3 & ③ & \end{array}$$

Substitute $x=3/4$ into ① or ② to get $y=-5/4$.

Elimination is the method that works well with more equations & in any dimension.

We're going to do a lot of Elimination!