MTH 210 1/21/2023

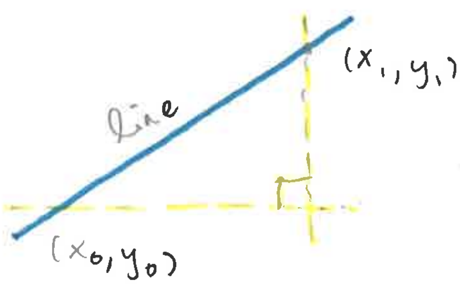
Lines & Planes

Describing lines in the plane ("the plane" $= \mathbb{R}^2$)

Old ways:

- slope & intercept: $y = \underset{\text{slope}}{m}x + \underset{\text{intercept}}{b}$
- point (x_0, y_0) & slope m: $(y - y_0) = m(x - x_0)$
- point (x_0, y_0) & point (x_1, y_1): $m = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{x_1 - x_0}$

$$\rightarrow (y - y_0) = \left(\frac{y_1 - y_0}{x_1 - x_0}\right)(x - x_0)$$



These ways leave out vertical lines! ($x = \text{const.}$)

Rewrite the last equation as: $(y - y_0)(x_1 - x_0) = (y_1 - y_0)(x - x_0)$

$$\rightarrow \underbrace{(x_1 - x_0)}_{a} y + \underbrace{(y_0 - y_1)}_{b} x = \underbrace{y_0(x_1 - x_0) - x_0(y_1 - y_0)}_{c}$$

$$\rightarrow \boxed{ax + by = c}$$ (a, b, c are constants)

Linear Equation in 2 unknowns (x, y)

Linear equation in n unknowns:

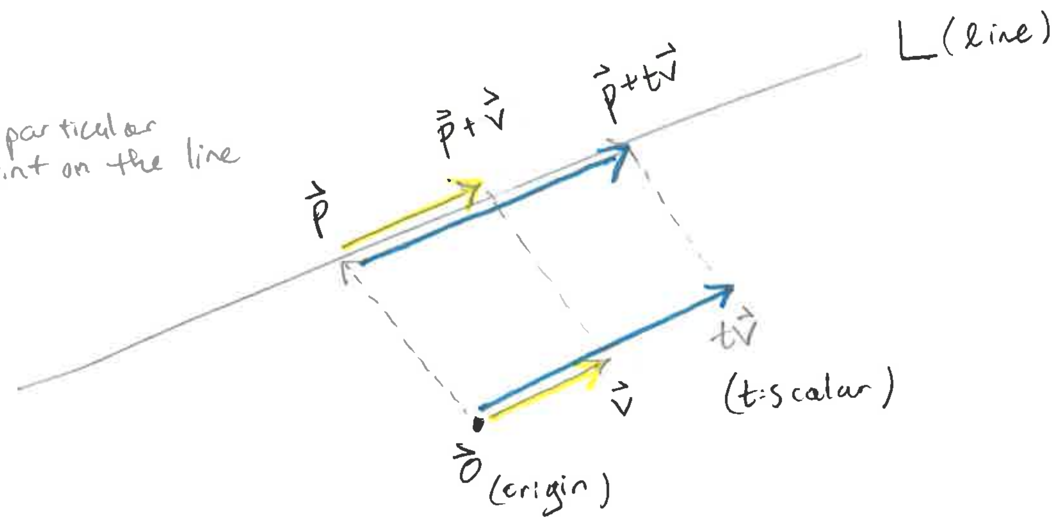
$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = c$$

where $a_1, a_2, \ldots, a_n$ and c are constants.

Describing lines using vectors:

(i) point $\vec{p}$ and direction vector $\vec{v}$

$\vec{p}$ = one particular point on the line

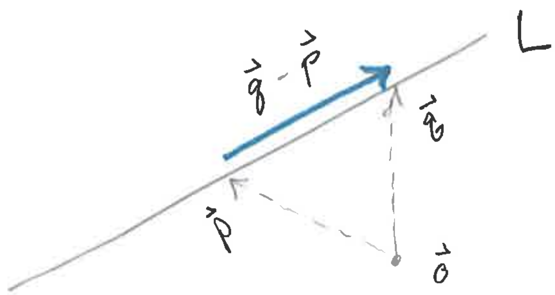


every point on the line L is the tip of an arrow for a vector of the form $\vec{v}+t\vec{p}$, t a scalar

$$\longrightarrow \quad L = \{\vec{p}+t\vec{v} \mid t \in \mathbb{R}\}$$

= "set of points of the form $\vec{p}+t\vec{v}$ where t is any real #"

(ii) Two points $\vec{p}, \vec{q}$



In this case:

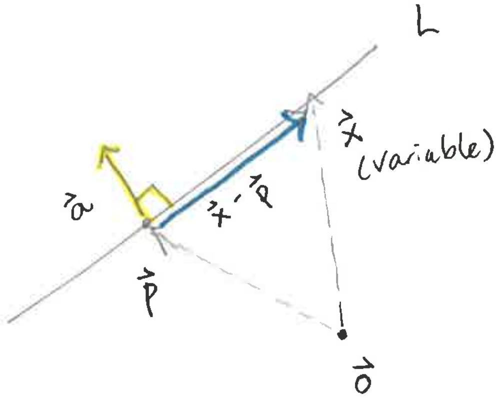
$$L = \{ \vec{p} + t(\vec{q}-\vec{p}) \mid t \in \mathbb{R} \}$$

$$= \{ (1-t)\vec{p} + t\vec{q} \mid t \in \mathbb{R} \}$$

$$= \{ s\vec{p} + t\vec{q} \mid s,t \in \mathbb{R},\ s+t=1 \}$$

(iii) point $\vec{p}$ & normal vector $\vec{a}$

$\underset{=}{\uparrow}$ perpendicular



symbol for "perpendicular"

$\vec{x}$ is on the line L $\iff$ $\vec{x}-\vec{p} \perp \vec{a}$

$\iff (\vec{x}-\vec{p})\cdot\vec{a} = 0$

$\vec{x}\cdot\vec{a} - \vec{a}\cdot\vec{p} = 0$

$\vec{x}\cdot\vec{a} = \vec{a}\cdot\vec{p}$

Note $\vec{a}\cdot\vec{p}$ is a constant scalar, call it c.

Then: $L = \{ \vec{x} \mid \vec{a}\cdot\vec{x} = c \}$.

Now suppose $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\vec{a} = \begin{bmatrix} a \\ b \end{bmatrix}$. Then $\vec{a}\cdot\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}\cdot\begin{bmatrix} x \\ y \end{bmatrix}$

$= ax+by$.

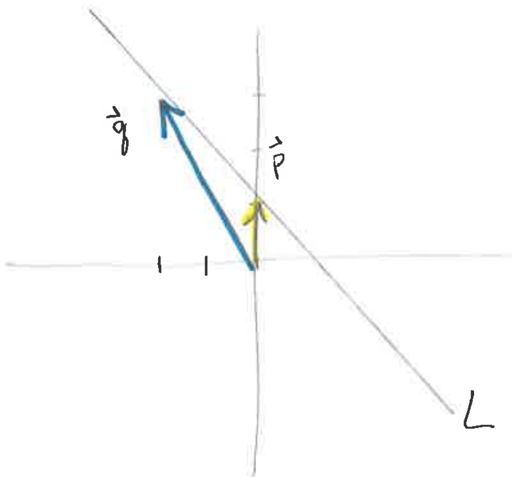
so $\vec{a}\cdot\vec{x} = c$ becomes $ax+by=c$

(same equation as earlier!)

Example

Let L be a line containing $\vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{q} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Express an equation for L in the form $\vec{a} \cdot \vec{x} = c$.



$L = \{ \vec{p} + t(\vec{q} - \vec{p}) \mid t \in \mathbb{R} \}$

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ be any point on L.

$\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$ for some scalar t

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2t \\ 1+2t \end{bmatrix}$

$\rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2t \\ 1+2t \end{bmatrix} \rightarrow \begin{cases} x = -2t \\ y = 1+2t \end{cases}$

Eliminate t: $\left. \begin{array}{l} t = -\frac{1}{2}x \\ t = \frac{1}{2}(y-1) \end{array} \right\} \rightarrow -\frac{1}{2}x = \frac{1}{2}(y-1) \rightarrow -x = y-1$

$\rightarrow x + y = \underbrace{1}_{c}$

Note $\underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\vec{a}} \cdot \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = (1)x + (1)y$. So $x+y=1$ is $\vec{a} \cdot \vec{x} = c$

where $\vec{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $c = 1$.

Faster, geometric way:

If $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ then $\vec{w} = \begin{bmatrix} v_2 \\ -v_1 \end{bmatrix}$ is $\perp$ to $\vec{v}$: $\vec{v} \cdot \vec{w} = v_1(v_2) + v_2(-v_1) = 0$.

L has direction vector $\vec{q} - \vec{p} = \begin{bmatrix} -2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

So $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is $\perp$ to $\vec{q} - \vec{p}$.

Therefore L has an equation $\vec{a} \cdot \vec{x} = c$

In fact $c = \vec{a} \cdot \vec{p} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 + 2 = 2$.

So $\vec{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $c = 2$ also work!

On to 3-dimensions...

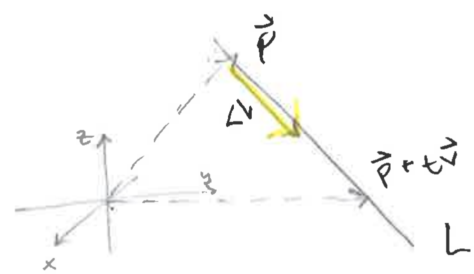
What equation describes a line in $\mathbb{R}^3$?

None! A line in $\mathbb{R}^3$ cannot be described by $\underline{1}$ equation.

How to describe lines in $\mathbb{R}^3$:

(i) point $\vec{p}$ & direction vector $\vec{v}$

$$L = \{ \vec{p} + t\vec{v} \mid t \in \mathbb{R} \}$$



(ii) point $\vec{p}$ & point $\vec{q}$ (also same as before)

$$L = \{ \vec{p} + t(\vec{q} - \vec{p}) \mid t \in \mathbb{R} \} = \{ s\vec{p} + t\vec{q} \mid s + t = 1 \}$$

But point $\vec{p}$ & normal vector $\vec{a}$ does not describe a line in $\mathbb{R}^3$

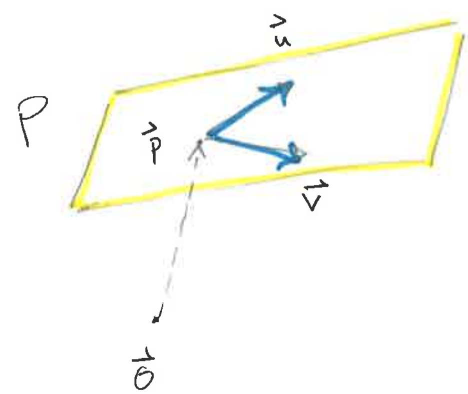
$\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ fixed $\qquad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ variable $\qquad d$ fixed scalar

$$\vec{a} \cdot \vec{x} = d \;\rightarrow\; \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = d \;\rightarrow\; \boxed{ax + by + cz = d}$$

this describes a plane in $\mathbb{R}^3$

Ways to describe planes in $\mathbb{R}^3$:

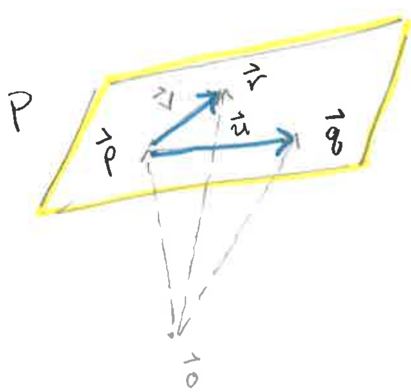
(i) point $\vec{p}$ & two direction vectors $\vec{u}, \vec{v}$



plane

$$P = \{ \vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R} \}$$

(ii) three points $\vec{p}, \vec{q}, \vec{r}$



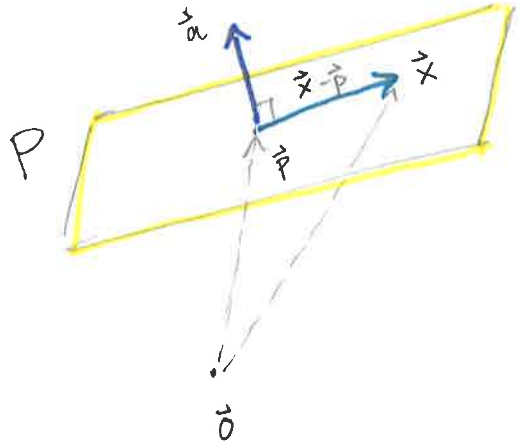
$\vec{v} = \vec{r} - \vec{p}$

$\vec{u} = \vec{q} - \vec{p}$

plane

$$\begin{aligned} P &= \{ \vec{p} + s\vec{u} + t\vec{v} \mid s,t \in \mathbb{R} \} \\ &= \{ \vec{p} + s(\vec{q} - \vec{p}) + t(\vec{r} - \vec{p}) \mid s,t \in \mathbb{R} \} \\ &= \{ (1-s-t)\vec{p} + s\vec{q} + t\vec{r} \mid s,t \in \mathbb{R} \} \\ &= \{ a\vec{p} + b\vec{q} + c\vec{r} \mid a+b+c=1 \} \end{aligned}$$

(iii) point $\vec{p}$ & normal vector $\vec{a}$



$\vec{x}$ any point on plane P satisfies

$$\vec{x}-\vec{p} \perp \vec{a} \iff (\vec{x}-\vec{p})\cdot\vec{a} = 0$$
$$\iff \vec{x}\cdot\vec{a} - \vec{x}\cdot\vec{p} = 0$$
$$\iff \vec{x}\cdot\vec{a} = \vec{x}\cdot\vec{p}$$

with $\vec{x}\cdot\vec{p} = d$, $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\vec{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ last equation becomes

$$\boxed{ax + by + cz = d}$$

in $\mathbb{R}^1$: $ax = b$ ... point (0-dim)

in $\mathbb{R}^2$: $ax + by = c$ ... line (1-dim)

in $\mathbb{R}^3$: $ax + by + cz = d$ ... plane (2-dim)

⋮

in $\mathbb{R}^n$: $a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ ... "hyperplane" ((n-1)-dim)

These are all of the form

$$\vec{a}\cdot\vec{x} = \text{constant}$$

where $\vec{a}$ is perpendicular (to line, plane, ...)

Algebra $\longleftrightarrow$ Geometry