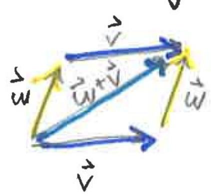
MTH 210 1/19/23 ①

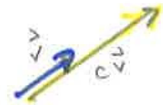
Last time:

vectors

adding

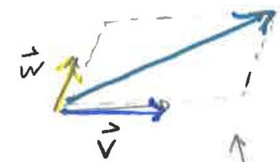


scaling



linear combinations

$c\vec{v}+d\vec{w}$



in $\mathbb{R}^2$, all linear combinations $c\vec{v}+d\vec{w}$ "typically" fill out $\mathbb{R}^2$

General linear combination of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$$

where $c_1, c_2, \dots, c_n$ are scalars.

In $\mathbb{R}^3$ (3-dimensional space):

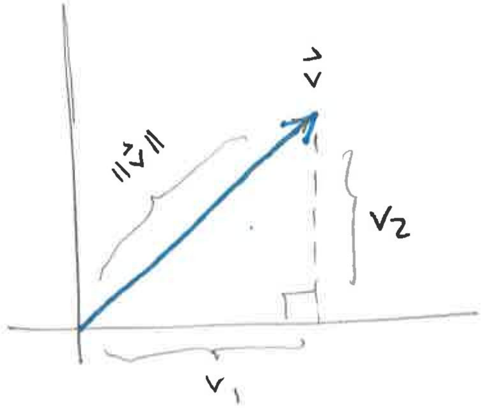
All linear combinations	"typically"	fill out a...
$c\vec{u}$		line
$c\vec{u}+d\vec{v}$		plane
$c\vec{u}+d\vec{v}+e\vec{w}$		all of $\mathbb{R}^3$

Linear combinations are very important — but let's go back to some basic geometry for now.

Lengths & Angles (§1.2)

Let's work in $\mathbb{R}^2$.

Given $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ let $\|\vec{v}\|$ denote the length of the arrow defining $\vec{v}$.

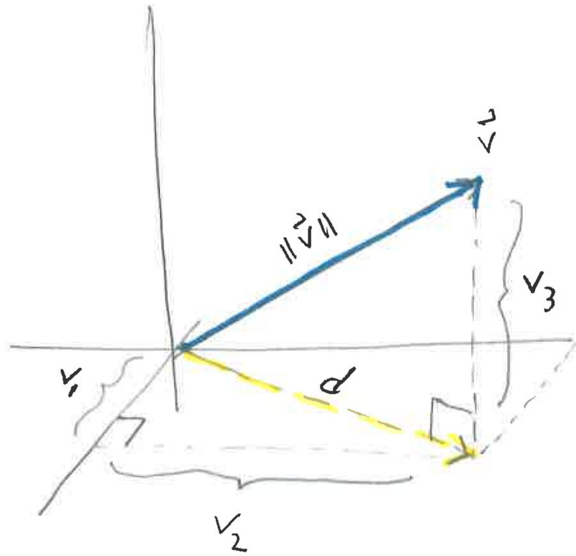


Pythagorean Theorem

$\rightarrow \quad \|v\|^2 = v_1^2 + v_2^2$

$\|v\| = \sqrt{v_1^2 + v_2^2}$

Now move to $\mathbb{R}^3$.



$d^2 = v_1^2 + v_2^2$

$\|\vec{v}\| = \sqrt{d^2 + v_3^2}$

$\rightarrow \quad \|v\|^2 = \sqrt{v_1^2 + v_2^2 + v_3^2}$

For $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ we define its length to be

$$\boxed{\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}}$$

Distance between two vectors $\vec{u}, \vec{v}$:

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\| = \sqrt{(u_1 - v_1)^2 + \dots + (u_n - v_n)^2}$$

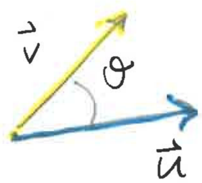
Example: Compute distance between vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{u} - \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\|\vec{u} - \vec{v}\| = \sqrt{1^2 + 1^2 + 0^2 + 1^2} = \sqrt{3}$. So distance is $\sqrt{3}$.

Angles



To understand angles between vectors we introduce the dot product:

for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$,

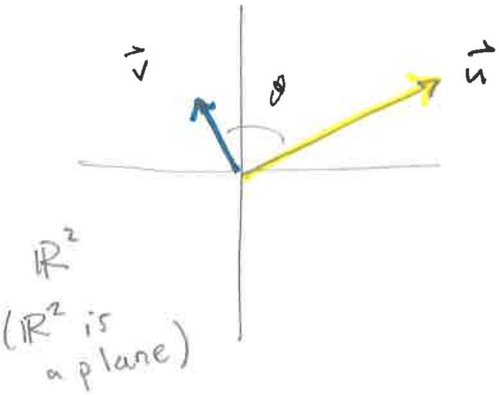
$$\boxed{\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n}$$

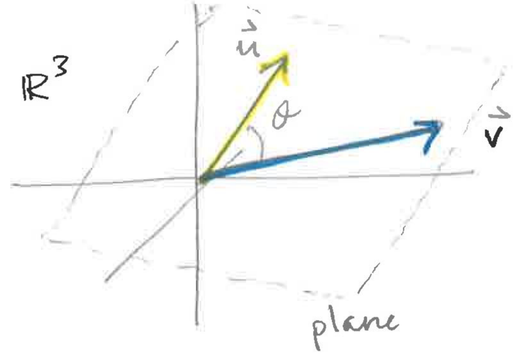
Note: (vector) $\cdot$ (vector) = scalar!

Also: $\|\vec{v}\|^2 = v_1^2 + v_2^2 + \dots + v_n^2 = v_1 v_1 + v_2 v_2 + \dots + v_n v_n = \vec{v} \cdot \vec{v}$

$$\longrightarrow \boxed{\|\vec{v}\|^2 = \vec{v} \cdot \vec{v}}$$

Given $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, they live in a 2D plane (regardless of n)





So it always makes sense to talk about the angle b/w $\vec u, \vec v$.

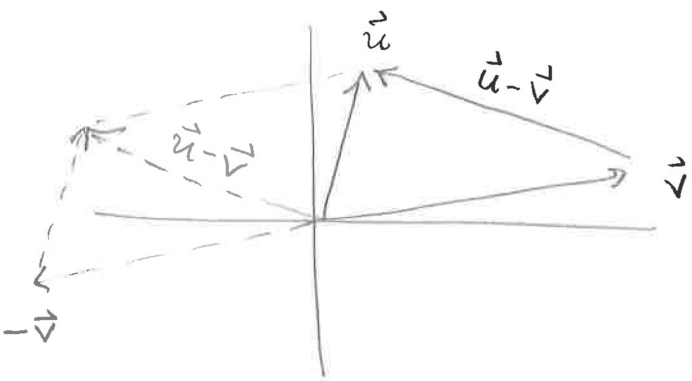
We claim: $\boxed{\vec u \cdot \vec v = \|\vec u\| \, \|\vec v\| \cos\theta}$

vectors   scalars

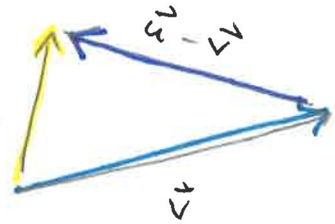
We'll need...

Geometric meaning of $\vec u - \vec v$:

$\vec u - \vec v = \vec u + (-\vec v)$

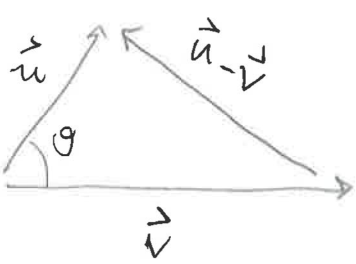


so:



$\vec u - \vec v$ is an arrow pointing from head of $\vec v$ to head of $\vec u$

Back to angles:



Law of cosines says:

$$\|\vec u - \vec v\|^2 = \|\vec u\|^2 + \|\vec v\|^2 - 2\|\vec u\| \, \|\vec v\| \cos\theta$$

Compute: $\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}) = \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

(use $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$) $\quad = \vec{u}\cdot\vec{u} - 2\vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$

$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}$$

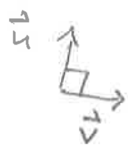
Substituting into left side of Law of cosines gives

$$\cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\vec{u}\cdot\vec{v} = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2} - 2\|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

$$\longrightarrow \vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta,$$

which is the formula we claimed.

Consequence: $\vec{u},\vec{v}$ are perpendicular $\iff \vec{u}\cdot\vec{v} = 0$.
($\theta = 90° = \frac{\pi}{2}$ rad)



this means "if and only if".

Example: Compute angle between $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$ (recall also $\|\vec{u}\|^2 = \vec{u}\cdot\vec{u}$)

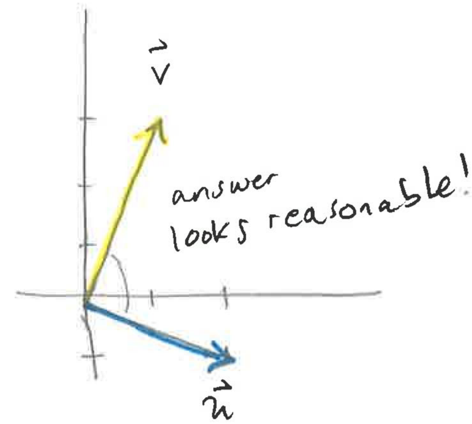
$$\vec{u}\cdot\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2^2 + (-1)^2 = 5$$

$$\vec{v}\cdot\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1^2 + 3^2 = 10$$

$$\vec{u}\cdot\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\cdot\begin{bmatrix} 1 \\ 3 \end{bmatrix} = (2)(1) + (-1)(3) = -1$$

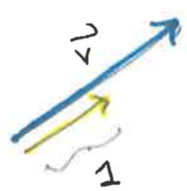
$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|} = \frac{-1}{\sqrt{5}\,\sqrt{10}} = \frac{-1}{5\sqrt{2}} \approx -0.14142$$

$$\theta \approx \arccos(-0.14142) \approx 98.13^\circ$$



<u>Unit vectors</u>: any vector $\vec{u}$ of length 1

Any "nonzero" vector $\vec{v}$ can be rescaled to a unit vector



Take $\dfrac{\vec{v}}{\|\vec{v}\|} = \underbrace{\left(\dfrac{1}{\|\vec{v}\|}\right)}_{\text{scalar}} \underbrace{\vec{v}}_{\text{vector}}$ ← this is a unit vector in direction of $\vec{v}$

Check it's of unit length:

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \left(\frac{1}{\|\vec{v}\|}\right)\vec{v} \right\| = \frac{1}{\|\vec{v}\|}\|\vec{v}\| = 1$$

(we're using that $\|c\vec{v}\| = |c|\,\|\vec{v}\|$ for a scalar c.)

<u>Example</u>: Find a unit vector perpendicular to $\vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. We want $\vec{u} \cdot \vec{v} = 0$.

$$0 = \vec{u} \cdot \vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2v_1 - v_2 \quad \rightarrow \quad 2v_1 = v_2$$

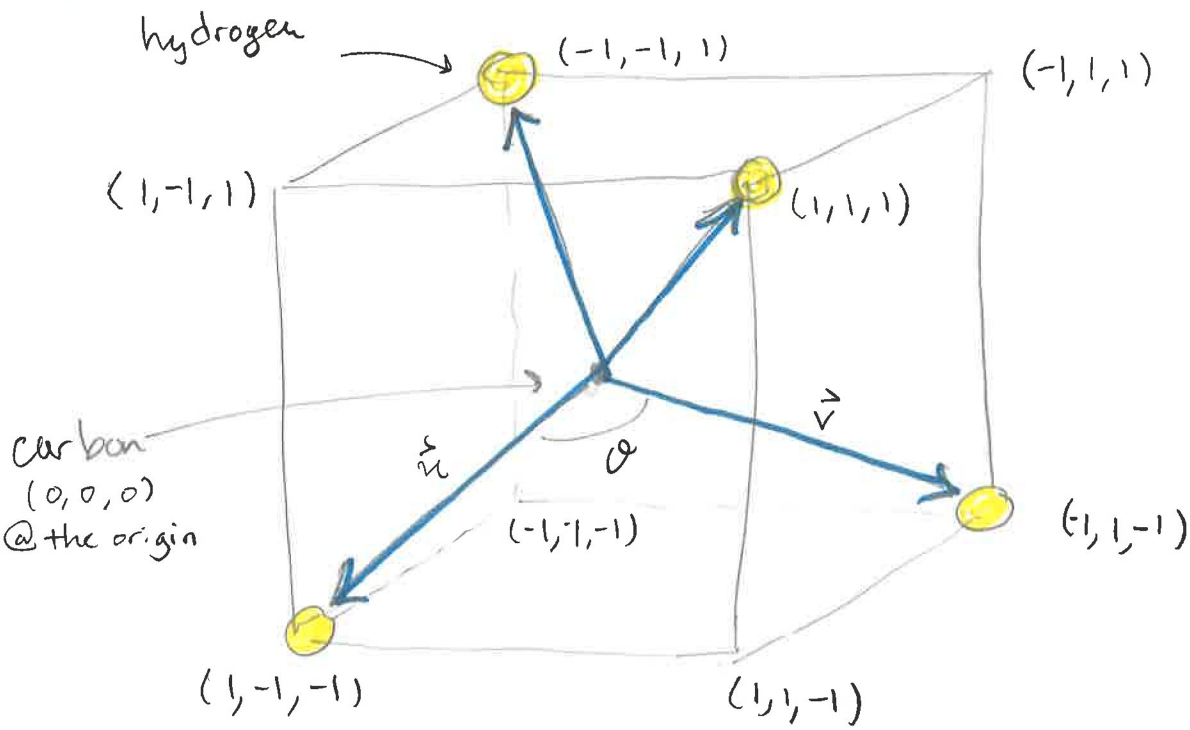
Can take $v_1 = 1$, $v_2 = 2$. So $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is perp. to $\vec{u}$.

To get a unit vector, take

$$\frac{\vec{v}}{\|\vec{v}\|}. \qquad \text{Note } \|\vec{v}\|^2 = \vec{v} \cdot \vec{v} = 1^2 + 2^2 = 5,$$

$$\text{so } \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$

__Example__ Find angle between two hydrogen atoms in a methane molecule CH_4.



$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{u} \cdot \vec{u} = 1^2 + (-1)^2 + (-1)^2 = 3 \longrightarrow \|\vec{u}\| = \sqrt{3}$$

$$\vec{v} \cdot \vec{v} = (-1)^2 + 1^2 + (-1)^2 = 3 \longrightarrow \|\vec{v}\| = \sqrt{3}$$

$$\vec{u} \cdot \vec{v} = (1)(-1) + (-1)(1) + (-1)(-1) = -1 - 1 + 1 = -1$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{-1}{\sqrt{3}\sqrt{3}} = \frac{-1}{3}$$

$$\theta = \arccos\left(-\frac{1}{3}\right) \approx 109.5^\circ.$$