MTH 210 - Linear Algebra

1/17/23

(Syllabus, webpage, etc.)

Vectors & Linear combinations

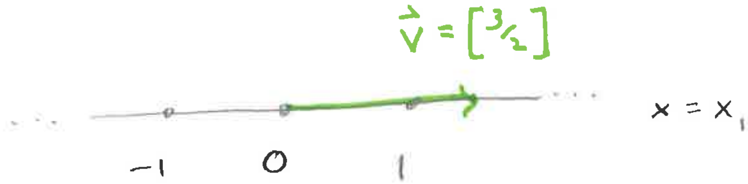
§1.1 in Strang

$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R} \}$$

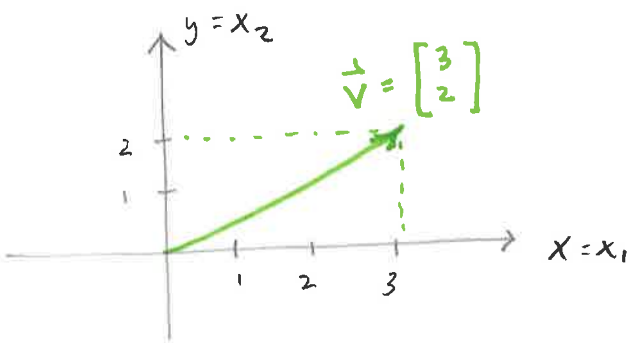
↑ n-dimensional Euclidean space

$\mathbb{R}$ = set of real #'s

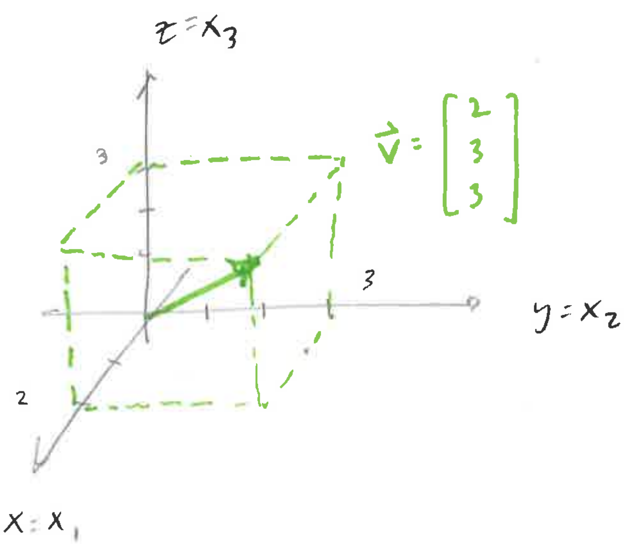
$\underline{n=1}$: $\mathbb{R}^1$



$\underline{n=2}$: $\mathbb{R}^2$



$\underline{n=3}$: $\mathbb{R}^3$

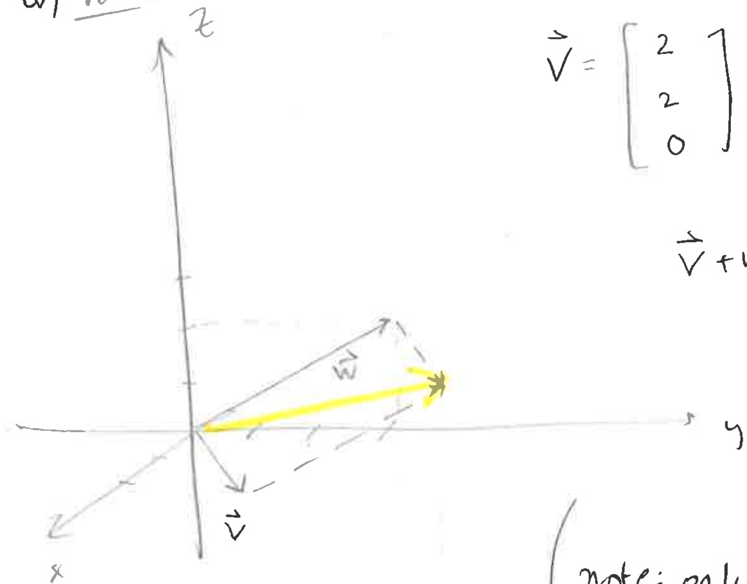


$\underline{n \geq 4}$: harder to visualize!

A $\underline{\text{vector}}$ $\vec{v}$ in $\mathbb{R}^n$ is an arrow starting from the origin + ending at some other pt.

Coordinate: $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$ $\quad \vec{v} + \vec{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}$

example w/ $n=3$:



$\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} 2+0 \\ 2+3 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$

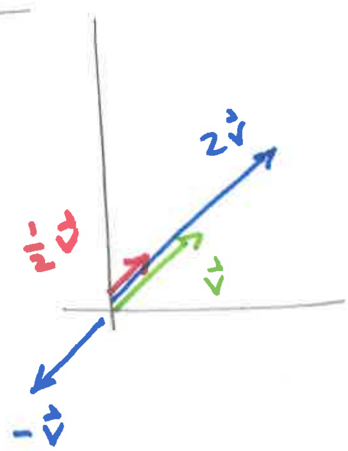
(note: only need to use two sides of parallelogram )

Scalar Multiplication

$\vec{v}$ vector in $\mathbb{R}^n$

c real # ("scalar") $\rightsquigarrow$ new vector $c\vec{v}$ in $\mathbb{R}^n$

Geometric:



$c > 0$:
Scale vector by c

$c = -1$: reverse vector through origin

$c < 0$:
reverse and scale by $|c|$

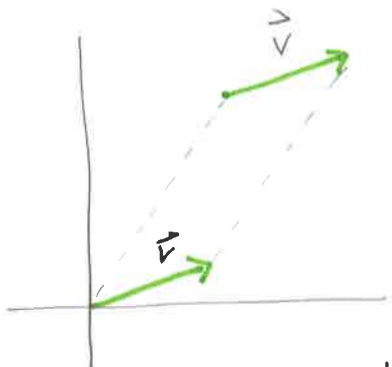
$c = 0$: $\quad 0\vec{v} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ zero vector

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ "column vector"

components

$(v_1, v_2, \dots, v_n)$ are the coordinates in $\mathbb{R}^n$ of where the arrow ends.

We may also consider arrows in $\mathbb{R}^n$ not necessarily emanating from the origin



we can always translate this vector so that it starts at the origin.

It is convenient to "equate" the two arrows: the original arrow is the same as the one starting at the origin.

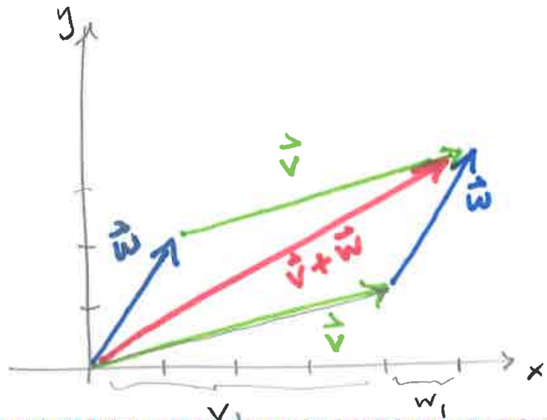
(so we write $\vec{v}$ for both).

Vector Addition

$\vec{v}, \vec{w}$ vectors in $\mathbb{R}^n$ $\rightsquigarrow$ "$\vec{v} + \vec{w}$", a vector in $\mathbb{R}^n$

Two descriptions:

Geometric:



$\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{v} + \vec{w} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Coordinate:

$$c\vec{v} = c\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ \vdots \\ cv_n \end{bmatrix}$$

A linear combination of two vectors $\vec{v}, \vec{w}$ is any vector of the form

$$c\vec{v} + d\vec{w}$$

where c, d are scalars.

Example Find all linear combinations of $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

c, d	$c\vec{v} + d\vec{w}$
$1, 0$	$1\vec{v} + 0\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$0, 1$	$0\vec{v} + 1\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
$1, -1$	$1\vec{v} + (-1)\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$
$2, 1$	$2\vec{v} + (1)\vec{w} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+2 \\ 2-1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

There are ∞ possibilities.

In fact: we get all vectors in $\mathbb{R}^2$.

Algebraic argument:

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be any vector in $\mathbb{R}^2$.

Can we find scalars c, d s.t. $\begin{bmatrix} x \\ y \end{bmatrix} = c\vec{v} + d\vec{w}$?

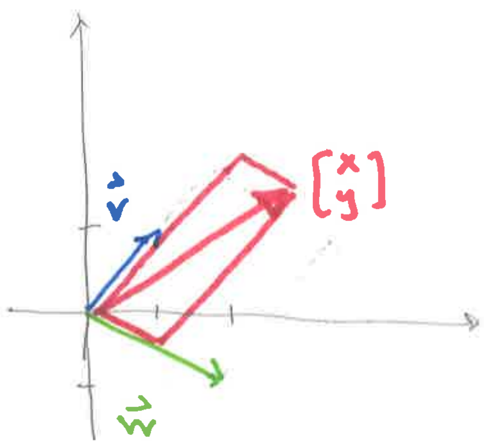
$$\begin{bmatrix} x \\ y \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} c+2d \\ c-d \end{bmatrix}$$

get 2 eqs: $\begin{cases} c+2d = x \\ c-d = y \end{cases}$

Subtract 1^{st} eq from 2^{nd}: $-3d = y - x \rightsquigarrow \boxed{d = \frac{x-y}{3}}$

Substitute d into 2^{nd} eq: $\boxed{c = \frac{x+2y}{3}}$

Geometric interp.:



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3/2 \end{bmatrix} = c\vec{v} + d\vec{w}$$

$$c = \frac{3 + 2(3/2)}{3} = \frac{3+3}{3} = 2$$

$$d = \frac{3 - 3/2}{3} = \frac{3/2}{3} = \frac{1}{2}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = 2\vec{v} + \frac{1}{2}\vec{w}$$

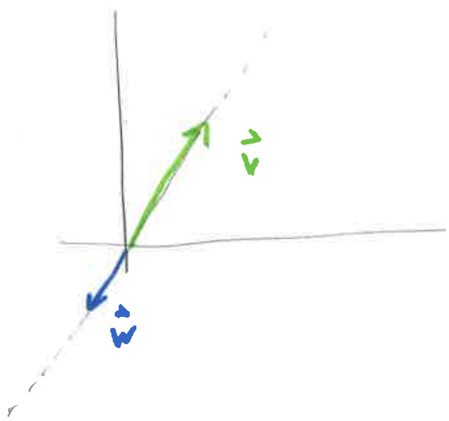
Geometric argument:

Any $\begin{bmatrix} x \\ y \end{bmatrix}$ is the far corner of a parallelogram whose sides are scalings of $\vec{v}$, $\vec{w}$.

Example: Consider linear combinations of $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix}$.

$$c\vec{v} + d\vec{w} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + d\begin{bmatrix} -\frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} c - d/2 \\ 2c - d \end{bmatrix}$$

All linear combs lie on the line $y = 2x$.



So contrary to previous example, we do not get all vectors in $\mathbb{R}^2$.

Linear combination of vectors $\vec{u}, \vec{v}, \vec{w}$:

$$c\vec{u} + d\vec{v} + e\vec{w}$$

where c, d, e are scalars

In 3-dimensions (in $\mathbb{R}^3$):

All linear combinations	"typically" fill out a...
$c\vec{u}$	line
$c\vec{u} + d\vec{v}$	plane
$c\vec{u} + d\vec{v} + e\vec{w}$	all of $\mathbb{R}^3$

General definition of linear combination:

A linear combination of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$

is a vector of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$

where $c_1, c_2, \ldots$ are scalars.