Homework 7

In this homework you will study projections and related problems.

Do the following problems:

A. Consider the plane in $\mathbb{R}^3$ spanned by $(1,-1,1)$ and $(-2,1,1)$. Find the 3×3 matrix P which projects vectors in $\mathbb{R}^3$ onto this plane. Using P, project the vector $(1,0,0)$ onto the plane.

B. Let A be a matrix with independent columns, and $P = A(A^TA)^{-1}A^T$ the associated projection matrix which projects to the column space of A.

(a) Show that $P^2 = P$.

(b) Show that $P^T = P$.

(c) Show the matrix $I - P$ also satisfies (a), (b): $(I-P)^2 = I - P$ and $(I-P)^T = I - P$.

(d) Give any vectors $\mathbf{x}$, $\mathbf{y}$ show that $(I-P)\mathbf{x}$ is orthogonal to $P\mathbf{y}$.

Commentary: For these computations, use the general properties $(BC)^T = C^TB^T$ and $(B^{-1})^T = (B^T)^{-1}$ and $(B+C)^T = B^T + C^T$, which you should convince yourself are true.

Any square matrix P satisfying $P^2 = P$ and $P^T = P$ is in fact the projection matrix onto some subspace! The subspace that P projects onto is simply $C(P)$, the column space of P. (Which happens to agree with $C(A)$, if P is first defined as $A(A^TA)^{-1}A^T$.)

Part (c) shows $I - P$ is a projection matrix. Part (d) shows that it projects onto a subspace of the orthogonal complement of $C(P)$. In fact, $I - P$ projects onto the whole of $C(P)^\perp$, not just a subspace.

C. Use (A) and (B) to compute the following: project $(0,1,0)$ onto the line which is orthogonal to the plane spanned by $(1,-1,1)$ and $(-2,1,1)$.

D. (Least squares problem) Let $(x,y) = (0,1),(1,0),(-1,1),(0,0)$ be four given points in $\mathbf{R}^2$. In this problem we find the parabola $y = ax^2 + bx + c$ that best approximates these points.

(a) Plug the four points into $y = ax^2 + bx + c$ to obtain four equations, with a, b, c unknowns.

(b) Letting $\mathbf{x}$ be the vector (a,b,c), write your equations of (a) in the form $A\mathbf{x} = \mathbf{b}$.

(c) Find the least squares solution: to do this, solve $A^TA\mathbf{x} = A^T\mathbf{b}$.

(d) Plot the points and the parabola in the plane.

E. The vectors $\mathbf{v}_1 = (1,0,1,0)$, $\mathbf{v}_2 = (0,1,1,0)$, $\mathbf{v}_3 = (0,0,-1,1)$ span a 3-dimensional subspace W in $\mathbf{R}^4$. Using the Gram–Schmidt method, find an orthonormal basis of W.