Homework 6

In this homework you will continue studying aspects of the Rank-Nullity Theorem.

Do the following problems:

A. Consider the following vectors in $\mathbb{R}^4$:

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let W be the subspace of $\mathbb{R}^4$ that is spanned by these vectors. Find a basis for W and compute the dimension of W.

B. Let A be an $m \times n$ matrix. Recall that the nullspace of A, written $N(A)$, is the subspace of $\mathbb{R}^n$ which consists of all solutions $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$. Thus $\dim N(A)$ is the dimension of the space of solutions to $A\mathbf{x} = \mathbf{0}$. Recall also that the column space of A, written $C(A)$, is the subspace of $\mathbb{R}^m$ spanned by the columns in A. Suppose that

$$C(A) = \mathbb{R}^m$$

i.e. the column space is "as large as possible". Show that under this assumption, the Rank-Nullity Theorem implies that the dimension of the space of solutions to $A\mathbf{x} = \mathbf{0}$ is equal to the "expected dimension" that we defined earlier in the course.[1]

C. Suppose A is an $m \times n$ matrix. Define a function $T : \mathbb{R}^n \to \mathbb{R}^m$ by sending any vector $\mathbf{v}$ in $\mathbf{R}^n$ to the vector $T(\mathbf{v}) = A\mathbf{v}$ in $\mathbb{R}^m$. Show that T is a linear transformation. (Recall from the notes: you must check that T satisfies two properties.)

D. Recall that the sum of two vector spaces $U, W \subset \mathbb{R}^n$ is defined to be

$$U + W = \{\mathbf{u} + \mathbf{w} \mid \mathbf{u} \text{ is in } U,\ \mathbf{w} \text{ is in } W\}$$

Show that this is a subspace of $\mathbb{R}^n$. (Verify the properties of being a subspace!)

E. Let V and W be subspaces of $\mathbb{R}^n$.

(i) If $n = 4$, $\dim V = 2$ and $\dim W = 2$, what are the possibilities for $\dim(V \cap W)$?

(ii) If $n = 7$, $\dim V = 4$ and $\dim W = 4$, what are the possibilities for $\dim(V \cap W)$?

[1] The condition $C(A) = \mathbb{R}^m$ is in fact very "common": for a random matrix A, it will hold.

F. In each case, determine whether the assignment $T : V \to W$ described is a linear transformation. In doing so, you should either verify that the two properties of being a linear transformation hold, or that they fail in some way.

(i) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = x^2$ for any real number x.

(ii) $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = 4x$ for any real number x.

(iii) $T : \mathbb{R} \to \mathbb{R}^2$ given $T(x) = (1, 5x - 1)$ for any real number x.

(iv) $T : \mathbb{R}^3 \to \mathbb{R}$ sends a vector to its first entry.