Rank three instantons, representations and sutures

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Abstract

We show that the knot group of any knot in any integer homology sphere admits a non-abelian representation into $SU(3)$ such that meridians are mapped to matrices whose eigenvalues are the three distinct third roots of unity. This answers the $N=3$ case of a question posed by Xie and the first author. We also characterize when a $PU(3)$-bundle admits a flat connection. The key ingredient in the proofs is a study of the ring structure of $U(3)$ instanton Floer homology of $S^1 \times \Sigma_g$. In an earlier paper, Xie and the first author stated the so-called eigenvalue conjecture about this ring, and in this paper we partially resolve this conjecture. This allows us to establish a surface decomposition theorem for $U(3)$ instanton Floer homology of sutured manifolds, and then obtain the mentioned topological applications. Along the way, we prove a structure theorem for $U(3)$ Donaldson invariants, which is the counterpart of Kronheimer and Mrowka's structure theorem for $U(2)$ Donaldson invariants. We also prove a non-vanishing theorem for the $U(3)$ Donaldson invariants of symplectic manifolds.

Contents

1 Introduction

This paper studies invariants in low-dimensional topology derived from $U(N)$ instanton gauge theory, with an emphasis on the case $N = 3$. Before describing the particular invariants and the general strategy, we begin with the central topological applications, which regard the existence of certain non-abelian representations of fundamental groups of 3-manifolds.

$U(3)$ representations of 3-manifold groups

Let $N \geqslant 2$ be an integer. The following was posed by the first author and Xie [DX20]:

Question 1.1. If K is a non-trivial knot in an integer homology 3-sphere Y, does there exist a homomorphism $\phi : \pi_1(Y\backslash K) \to SU(N)$ with non-abelian image, such that

$$\phi(\mu) = c \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \zeta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \zeta^{N-1} \end{bmatrix}$$

where $\zeta = e^{2\pi i/N}$ and $c = e^{\pi i/N}$ or $c = 1$ depending on whether N is even or odd?

The notation μ refers to the class of a fixed meridian in the knot group. Note that if Question 1.1 has an affirmative answer for N, then it does so too for all lN, where $l \in \mathbf{Z}_{>0}$. Kronheimer and Mrowka proved that Question 1.1 has an affirmative answer in the case $N = 2$ [KM10b]. In this paper we answer it affirmatively in the case $N = 3$:

Theorem 1.2. *If K is a non-trivial knot in an integer homology 3-sphere Y, then there exists a homomorphism $\phi : \pi_1(Y\backslash K) \to SU(3)$ with non-abelian image, such that*

$$\phi(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{bmatrix}, \qquad \zeta = e^{2\pi i/3}.$$

We also address the existence of 3-dimensional representations for fundamental groups of closed 3-manifolds. The following is an $N = 3$ analogue of a result of Kronheimer and Mrowka [KM10b, Thm. 7.21] (see also [KM18, Thm. 1.6]).

Theorem 1.3. *Let Y be a closed, oriented 3-manifold, and $\omega \in H^2(Y; \mathbf{Z}/3)$. Suppose $\omega[S] \equiv 0 \pmod 3$ for every embedded 2-sphere $S \subset Y$. Then there exists a homomorphism $\pi_1(Y) \to PU(3)$ with the associated characteristic class in $H^2(Y; \mathbf{Z}/3)$ equal to ω.*

The *Covering Conjecture* states that the N-fold branched cover $\Sigma_N(K)$ of a non-trivial knot K in a homotopy sphere Y is not a homotopy sphere [Kir78, Problem 3.38]. Theorem 1.2 provides a homomorphism ϕ which descends to a non-trivial homomorphism of $\pi_1(\Sigma_3(K))$, and thus proves the Covering Conjecture for $N = 3$. This also proves the Smith

Conjecture, for $N = 3$: a non-trivial knot is not the fixed point set of an order N orientation preserving diffeomorphism of S^3. Both the Covering Conjecture and Smith Conjecture for general N are theorems, proved by ideas and techniques from diverse areas of mathematics including hyperbolic geometry, minimal surface theory, $SL(2, \mathbf{C})$ character varieties and classical 3-manifold topology [Mor84]. The proof indicated for $N = 3$ (modelled on the proof of Kronheimer and Mrowka for $N = 2$) is based on instanton Floer theory.

The Floer homology we utilize is in the setting of $U(3)$ instanton gauge theory. Donaldson-type invariants for closed 4-manifolds can be defined for any of the groups $U(N)$, see [Kro05, Cul14]. (More precisely, the relevant gauge symmetry group is $PU(N)$.) Such invariants were first studied in the physics literature by Mariño and Moore [MM98]. There, a generalization of Witten's conjecture is provided, which predicts that no new topological information can be derived for 4-manifolds of simple type when $N \geqslant 3$. In contrast, Theorems 1.2 and 1.3 are derived from higher rank instanton Floer theory and do not follow from the $U(2)$ theory. To the best of the authors' knowledge, these theorems are the first genuine topological applications of higher rank instanton gauge theory.

$U(3)$ sutured instanton homology

Kronheimer and Mrowka proved analogues of the above results in the case $N = 2$ using $U(2)$ sutured instanton Floer homology [KM10b]. The strategy to address Question 1.1 in general is to develop sutured instanton Floer theory for $U(N)$ so that the proofs for the $N = 2$ case may be adapted. This was initiated in [DX20], where $U(3)$ sutured instanton homology was constructed and some basic properties were established. To a balanced sutured manifold (M, α), the construction outputs a $\mathbf{Z}/2$-graded complex vector space

$$SHI^3_*(M, \alpha). \tag{1.4}$$

This is done by first gluing $[-1, 1] \times F$ to M, where F is a connected surface of genus g with boundary; the gluing is such that $[-1, 1] \times \partial F$ is identified with annuli neighborhoods of the sutures $\alpha \subset \partial M$. As (M, α) is balanced, the resulting 3-manifold has two boundary components which are diffeomorphic, and gluing these up by a diffeomorphism forms a closed 3-manifold. Then (1.4) is defined by taking a certain subspace of the $U(3)$ instanton homology of the closed 3-manifold with some choice of admissible bundle.

In [DX20], it is shown that $SHI^3_*(M, \alpha)$ is independent of the gluing maps involved in the construction. Here we establish that the construction is also independent of the choice of F (in particular, the genus g). We obtain the following.

Theorem 1.5. *The $U(3)$ sutured instanton homology $SHI^3_*(M, \alpha)$ is independent of all auxiliary choices and is an invariant of the balanced sutured manifold (M, α).*

We also establish a *surface decomposition result*, which describes the behavior of $U(3)$ sutured instanton homology under surface decompositions. This is the counterpart of analogous results in sutured Heegaard Floer homology [Juh06] and $U(2)$ sutured instanton homology [KM10b]. The surface decomposition result, given in Proposition 5.16, leads to the following non-vanishing result, which (together with a modest generalization given in Corollary 5.19) is used to prove Theorems 1.2 and 1.3.

Theorem 1.6. *For any balanced taut sutured manifold (M, α), the $U(3)$ sutured instanton homology group $SHI^3_*(M, \alpha)$ is non-trivial.*

There are two important special cases of $U(3)$ sutured instanton homology. Both can be defined more generally in the setting of $U(N)$ instanton homology. The first is the $U(N)$ framed instanton homology for closed 3-manifold Y, denoted $I^{\#,N}(Y)$. Versions of these groups were first studied by Kronheimer and Mrowka in [KM11b]. Here we study some further basic properties. We compute in Theorem 8.4 that the Euler characteristic is

$$\chi\left(I^{\#,N}(Y)\right) = |H_1(Y;\mathbf{Z})|^{N-1} \tag{1.7}$$

when $b_1(Y) = 0$, and is otherwise zero. This generalizes the $N = 2$ computation from [Sca15]. Moreover, in the $N = 3$ case, we give a decomposition result for cobordism maps in framed instanton homology analogous to the $N = 2$ result in [BS23, Theorem 1.16], and which relies on an adaptation of the $U(3)$ Structure Theorem given below.

The other Floer homology group of interest is the $U(N)$ knot instanton homology for a knot in an integer homology 3-sphere. In the case $N = 2$, the graded Euler characteristic of this homology is a multiple of the Alexander polynomial [KM10a, Lim10]. For $N = 3$, we provide in Section 9 a conjectural relationship between the bi-graded Euler characteristic of the $U(3)$ knot homology and the Alexander polynomial, relying in part on the generalized version of Witten's conjecture from [MM98].

$U(3)$ Donaldson-type invariants for 4-manifolds

We also study the structure of $U(3)$ polynomial invariants for closed 4-manifolds. For any closed connected oriented smooth 4-manifold X define

$$\mathbf{A}^3(X) := (\mathrm{Sym}^*(H_0(X) \otimes H_2(X)) \otimes \Lambda^* H_1(X))^{\otimes 2} \tag{1.8}$$

where complex coefficients are assumed. For $\alpha \in H_i(X)$ where $i \in \{0, 1, 2\}$, and $r \in \{2, 3\}$, we write $\alpha_{(r)}$ when regarding α as an element of the $(r-1)^{\text{st}}$ tensor power of (1.8). The degree of $\alpha_{(r)}$ in this case is defined to be $2r - i$. If $b^+(X) > 1$, then for $w \in H^2(X;\mathbf{Z})$ there is an associated $U(3)$ Donaldson-type invariant

$$D^3_{X,w} : \mathbf{A}^3(X) \to \mathbf{C}.$$

Let $x \in X$, viewed as a generator of $H_0(X)$. We say that X is $U(3)$ *simple type* if

$$D^3_{X,w}(x^3_{(2)} z) = 27 D^3_{X,w}(z), \qquad D^3_{X,w}(x_{(3)} z) = 0, \qquad D^3_{X,w}(\delta z) = 0 \tag{1.9}$$

for all $z \in \mathbf{A}^3(X)$ and $\delta \in \Lambda^* H_1(X) \otimes \Lambda^* H_1(X)$. When $b_1(X) = 0$, this terminology is introduced in [DX20], and it is an analogue of Kronheimer and Mrowka's simple type condition in the $U(2)$ case; without the constraint on $b_1(X)$, it is an analogue of Muñoz's *strong simple type* condition [Muñ00]. Define for all $z \in \mathbf{A}^3(X)$:

$$\widehat{D}^3_{X,w}(z) := D^3_{X,w}((1 + \frac{1}{3}x_{(2)} + \frac{1}{9}x^2_{(2)})z).$$

It is also convenient to introduce the following formal power series in $\mathbf{C}[\![H_2(X) \oplus H_2(X)]\!]$:

$$\mathbb{D}^3_{X,w}(z) = \widehat{D}^3_{X,w}(e^z).$$

Our main result regarding the structure of these invariants is the following analogue of Kronheimer and Mrowka's structure theorem in the $U(2)$ case [KM95]. Let $\zeta = e^{2\pi i/3}$.

Theorem 1.10. *Suppose $b^+(X) > 1$, and X is $U(3)$ simple type. Then there is a finite set $\{K_i\} \subset H^2(X;\mathbf{Z})$ and $c_{i,j} \in \mathbf{Q}[\sqrt{3}]$ such that for any $w \in H^2(X;\mathbf{Z})$, and $\Gamma, \Lambda \in H_2(X)$:*

$$\mathbb{D}^3_{X,w}(\Gamma_{(2)} + \Lambda_{(3)}) = e^{\frac{Q(\Gamma)}{2} - Q(\Lambda)} \sum_{i,j} c_{i,j} \zeta^{w \cdot \left(\frac{K_i - K_j}{2}\right)} e^{\frac{\sqrt{3}}{2}(K_i + K_j)\cdot\Gamma + \frac{\sqrt{-3}}{2}(K_i - K_j)\cdot\Lambda} \quad (1.11)$$

Each class K_i is an integral lift of $w_2(X)$, and satisfies the following: if $\Sigma \subset X$ is a smoothly embedded surface of genus g with $\Sigma \cdot \Sigma \geqslant 0$ and $[\Sigma]$ non-torsion, then

$$2g - 2 \geqslant |\langle K_i, \Sigma\rangle| + [\Sigma]^2. \quad (1.12)$$

This result partially resolves Conjecture 7.2 from [DX20]. We note that the expression (1.11) differs slightly from what appears in that reference, due to a minor difference in convention; see Remark 2.3. As explained in [DX20], it is predicted by Mariño and Moore [MM98] that the classes K_i appearing in Theorem 1.10 are equal to the Kronheimer and Mrowka basic classes in $U(2)$ Donaldson theory, as well as the Seiberg–Witten basic classes; furthermore, the constants $c_{i,j}$ are expressible in terms of the data from these other theories.

Remark 1.13. In Theorem 6.26, if $[\Sigma]$ is torsion and $g \geqslant 1$, then (1.12) trivially holds. Note that if Σ is as in Theorem 1.10 and has genus zero, then (1.12) never holds, and hence there are no such classes K_i, in which case the invariants $D_{X,w}$ all vanish.

We also prove a non-vanishing result for symplectic 4-manifolds.

Theorem 1.14. *Let X be a closed symplectic 4-manifold with $b^+(X) > 1$. Then the invariant $D^3_{X,w}$ is non-trivial for all $w \in H^2(X;\mathbf{Z})$.*

Our strategy to prove this non-vanishing result is similar to Ozsváth and Szabó's proof in [OS04] for the corresponding result in the context of Heegaard Floer homology and closely follows a strategy suggested by Kronheimer and Mrowka in the $U(2)$ case. (The non-triviality of $U(2)$ Donaldson invariants for symplectic 4-manifolds was also proved by Sivek in a different way [Siv15].) Theorem 1.14 can be used to prove that the $U(3)$ instanton homology of an irreducible 3-manifold with 3-admissible bundle is non-zero, and leads to an alternative proof of Theorem 1.3.

Eigenvalues and the $U(3)$ instanton homology of $S^1 \times \Sigma_g$

The main technical ingredient that paves the way for most of the above results is Theorem 2.14 below, which concerns the $U(3)$ instanton homology of a circle times a surface Σ_g of genus g with an admissible bundle. We restrict our attention to the *simple type ideal*, a subspace of this Floer homology, and compute the eigenvalues of certain operators acting on it. The simple type ideal is generated by relative invariants coming from 4-manifolds of simple type. Our eigenvalue result is a partial analogue to one used by Kronheimer and Mrowka in the $U(2)$ case, due to Muñoz [Muñ99].

The $U(N)$ instanton Floer homology of $S^1 \times \Sigma_g$ with an admissible bundle is isomorphic to $H^*(\mathcal{N}_g)$, the cohomology of the moduli space of rank N stable holomorphic bundles over Σ_g with some fixed determinant. In fact, this instanton Floer group admits a multiplication which is a deformation of the cup product on $H^*(\mathcal{N}_g)$, and is expected to be isomorphic to its quantum multiplication. Muñoz's computation of eigenvalues in the $N = 2$ case relies on the fact that $H^*(\mathcal{N}_g)$ has a simple ring presentation which is recursive in the genus [Bar94, KN98, ST95, Zag95]. Such a concise description is not currently available in the $N = 3$ case, but a complete set of relations for the ring is known, due to Earl [Ear97]. Our restriction to the simple type ideal (which suffices for the purposes of the above results) simplifies the algebra considerably, and allows us to use Earl's description of the ring $H^*(\mathcal{N}_g)$ to prove, together with results from [DX20], the desired eigenvalue result.

The authors expect that the method of proof for Theorem 2.14 may also be employed for $N > 3$. There are two essential ingredients that are required. One is a generalization to $N \geqslant 4$ of [DX20, Prop. 5.7], which gives the existence of certain eigenvalues in the $U(N)$ instanton Floer homology of a circle times a surface. The other is a computation, for $N \geqslant 4$, of the vector space dimension of the ring $H^*(\mathcal{N}_g)$ modulo the "undeformed simple type relations," analogous to what is done below for $N = 3$. Relevant to this second ingredient is the work of Earl and Kirwan [EK04]. Given an appropriate generalization of Theorem 2.14 for $N > 3$, the authors expect that analogues for all of the results stated in this introduction, for general N, can also be proved, following similar methods. The authors hope to return to these matters in future work.

Outline In Section 2, we state and outline the proof of the main technical result, Theorem 2.14. In Sections 3 and 4, the cohomology ring $H^*(\mathcal{N}_g)$ is studied, and the proof of Theorem 2.14 is completed. In Section 5, we study $U(3)$ sutured instanton homology and prove Theorems 1.5, 1.6, followed by the proofs of Theorems 1.2 and 1.3. In Section 6, we prove Theorem 1.10, and in Section 7 we prove Theorem 1.14. In Section 8, $U(N)$ framed instanton homology is studied. Finally, in Section 9 we discuss $U(3)$ instanton knot homology and the Alexander polynomial.

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2 Background and general strategy

As mentioned in the introduction, the strategy to prove Theorems 1.2 and 1.3 is to develop sutured instanton Floer homology for the gauge group $U(N)$, and adapt arguments from the $N = 2$ case due to Kronheimer and Mrowka [KM10b]. This strategy was initiated in the $N = 3$ case by the first author and Xie [DX20]. The raw material for the construction of sutured instanton homology for general N is the $U(N)$ instanton Floer homology

$$I_*^N(Y, \gamma) \tag{2.1}$$

which is defined for a closed, oriented, connected 3-manifold Y and an oriented 1-cycle γ satisfying the N-*admissibility* condition: there exists some oriented surface $\Sigma \subset Y$ such that $\gamma \cdot \Sigma$ is coprime to N. The group $I_*^N(Y, \gamma)$ is constructed by applying Morse homological methods to a Chern–Simons functional on $\mathscr{B}$, the configuration space of connections on the $PU(N)$-bundle on Y determined by γ. These Floer homology groups were constructed by Kronheimer and Mrowka [KM11b], generalizing the work of Floer in the $N = 2$ case [Flo95]. In this paper, we work with Floer homology over the coefficient field $\mathbf{C}$, in which case (2.1) is a $\mathbf{Z}/4N$-graded complex vector space.

Given a homology class $a \in H_i(Y; \mathbf{C})$ there are associated linear operators

$$\mu_r(a) : I_*^N(Y, \gamma) \to I_*^N(Y, \gamma), \qquad 2 \leqslant r \leqslant N. \tag{2.2}$$

The degree of $\mu_r(a)$ is $2r - i \pmod{4N}$. There is a universal $PU(N)$-bundle $\mathbf{P}$ over $\mathscr{B} \times Y$, and $\mu_r(a)$ is roughly the cap product on moduli spaces with $c_r(\mathbf{P})/a$.

Remark 2.3. Our convention for $\mu_r(a)$ differs from that of [DX20] by the sign $(-1)^r$. Furthermore, the grading we use on instanton homology is the negative of the convention in that paper (and is in fact a cohomological grading convention).

The construction of sutured instanton Floer homology relies on taking certain simultaneous generalized eigenspaces of the operators in (2.2) acting on the Floer groups (2.1). The crucial case to understand is when $Y = S^1 \times \Sigma_g$ where Σ_g is a surface of genus g, and $\gamma = \gamma_d = S^1 \times \{x_1, \ldots, x_d\}$, with d coprime to N. We write

$$V_{g,d}^N := I_*^N(S^1 \times \Sigma_g, \gamma_d).$$

The relevant operators acting on $V_{g,d}^N$ are denoted as follows:

$$\alpha_r := \mu_r(\Sigma_g), \qquad \beta_r := \mu_r(x), \qquad \psi_r^i := \mu_r(\eta_i) \qquad (2 \leqslant r \leqslant N) \tag{2.4}$$

where $x \in S^1 \times \Sigma$ and the η_i $(1 \leqslant i \leqslant 2g)$ range over a symplectic basis of closed oriented curves on Σ_g. These operators are graded-commutative. In particular, since the ψ_r^i are of odd degree, they square to zero, and each one has zero as its only eigenvalue. Consider the simultaneous eigenvalues with respect to the classes α_r, β_r:

$$\Xi_{g,d}^N := \left\{ \lambda = (\lambda_1, \ldots, \lambda_{2N-2}) \in \mathbf{C}^{2N-2} \;\middle|\; \exists v \in V_{g,d}^N : \; \alpha_r v = \lambda_{r-1} v, \;\; \beta_r v = \lambda_{r+N-2} v \right\}$$

where r ranges over $2, \ldots, N$. For $\lambda \in \Xi_{g,d}^N$ we denote by

$$V_{g,d}^N(\lambda) = \bigcap_{r=2}^{N} \bigcup_{k=1}^{\infty} \ker\left((\alpha_r - \lambda_{r-1})^k\right) \cap \ker\left((\beta_r - \lambda_{r+N-2})^k\right) \subset V_{g,d}^N$$

the associated generalized eigenspace. Then we have

$$V_{g,d}^N = \bigoplus_{\lambda \in \Xi_{g,d}^N} V_{g,d}^N(\lambda).$$

Lemma 2.5. *Let ζ be a $2N^{\text{th}}$ root of unity. If $\lambda = (\lambda_1, \ldots, \lambda_{2N-2}) \in \Xi_{g,d}^N$, then also*

$$\lambda' := (\zeta\lambda_1, \zeta^2\lambda_2, \ldots, \zeta^{N-1}\lambda_{N-1}, \zeta^2\lambda_N, \zeta^3\lambda_{N+1}, \ldots, \zeta^{N-1}\lambda_{2N-3}, \zeta^N\lambda_{2N-2}) \in \Xi_{g,d}^N.$$

Furthermore, $\dim V_{g,d}^N(\lambda) = \dim V_{g,d}^N(\lambda')$.

Proof. Fix ζ as in the statement, and define $f : V_{g,d}^N \to V_{g,d}^N$ as follows. Let $v \in V_{g,d}^N$ and write v_i for the component of v in grading $i \pmod{4N}$. Then

$$f(v) := \sum_{i=0}^{2N-1} \zeta^{-i} v_{2i} + \sum_{i=0}^{2N-1} \zeta^{-i} v_{2i+1}.$$

Let $v \in V_{g,d}^N(\lambda)$. In particular, for each $2 \leqslant r \leqslant N$ we have $(\alpha_r - \lambda_{r-1})^N v = 0$ for some positive integer N. This is equivalent to the collection of identities

$$\sum_{i=0}^{N} \binom{N}{i} \alpha_r^{N-i} (-\lambda_{r-1})^i v_{l-(N-i)(2r-2)} = 0$$

where $2 \leqslant r \leqslant N$ and $0 \leqslant l \leqslant 4N-1$. It is straightforward to check that this identity is preserved upon replacing λ_{r-1} with $\zeta^{r-1}\lambda_{r-1}$ and replacing $v_{l-(N-i)(2r-2)}$ with $f(v)_{l-(N-i)(2r-2)}$. The conditions involving the β_r operators is similar. Thus f induces a vector space isomorphism from $V_{g,d}^N(\lambda)$ to $V_{g,d}^N(\lambda')$. $\square$

There is also an operator of degree $-4d \pmod{4N}$ denoted

$$\varepsilon : V_{g,d}^N \to V_{g,d}^N \tag{2.6}$$

defined as the map associated to the cylinder cobordism $[0,1] \times S^1 \times \Sigma_g$ equipped with $U(N)$-bundle determined by the oriented 2-cycle $[0,1] \times \gamma_d \cup \{(1/2, x)\} \times \Sigma_g$ where $x \in S^1$. The operator ε commutes with all the operators (2.4).

The Floer homology $V_{g,d}^N$ is in fact a ring. The multiplication is induced by the cobordism which is the product of a pair of pants cobordism $S^1 \sqcup S^1 \to S^1$ with Σ_g. An identity element $\mathbf{1}$ is given by the relative invariant $D^2 \times \Sigma_g$ with bundle determined by $D^2 \times \{x_1, \ldots, x_d\}$. Sending each operator (2.4) and ε to its evaluation on $\mathbf{1}$ induces an isomorphism of rings

$$V_{g,d}^N = \mathbf{A}_g^N[\varepsilon]/J_{g,d}^N \tag{2.7}$$

where the $\mathbf{Z}$-graded $\mathbf{C}$-algebra $\mathbf{A}_g^N$ is defined as follows:

$$\mathbf{A}_g^N := \bigotimes_{r=2}^{N} \mathbf{C}[\alpha_r, \beta_r] \otimes \Lambda^*(\psi_r^i)_{1 \leqslant i \leqslant 2g}$$

The degrees of $\alpha_r, \beta_r, \psi_r^i$ are respectively $2r-2, 2r, 2r-1$. In the identification (2.7), the $\mathbf{Z}$-grading on $\mathbf{A}_g^N$ reduced to the $\mathbf{Z}/4$-grading on $V_{g,d}^N$; on the right side of (2.7), the element ε should be regarded as having degree 0. The ideal $J_{g,d}^N \subset \mathbf{A}_g^N[\varepsilon]$ contains $\varepsilon^N - 1$ and is homogeneous with respect to the $\mathbf{Z}/4$-grading. There is a non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : V_{g,d}^N \otimes V_{g,d}^N \to \mathbf{C} \tag{2.8}$$

which is induced by $[0,1] \times S^1 \times \Sigma_g$ viewed as a cobordism from two copies of $S^1 \times \Sigma_g$ (identifying one copy by an orientation-reversing diffeomorphism) to the empty set.

Key to the development of Kronheimer and Mrowka's sutured instanton homology in the case $N = 2$ are results on the eigenvalues of the operators (2.4). There are two essential ingredients that are used, both following from the work of Muñoz [Muñ99] (see in particular [KM10b, Props. 7.1, 7.4]). The first is the computation of the spectrum:

$$\Xi_{g,1}^2 = \{(2ai^r, (-1)^r 2) \mid a \in \mathbf{Z},\ |a| \leqslant g-1,\ r \in \{0,1\}\} \tag{2.9}$$

where $i = \sqrt{-1}$. The second ingredient regards "extremal" generalized eigenspaces:

$$\dim V_{g,1}^2(\pm i^r(2g-2), (-1)^r 2) = 1. \tag{2.10}$$

An important property is that the pairing (2.8) restricted to the 1-dimensional space appearing in (2.10) is non-degenerate. In fact, the 1-dimensionality is equivalent to non-degeneracy, see for example [DX20, Lemma 5.11].

In [DX20], analogous properties for the case of $N = 3$ are studied. To state the relevant results, first define, for any integer d coprime to 3:

$$\mathcal{E}_{g,d}^3 := \left\{(\sqrt{3}\zeta^k a, \sqrt{-3}\zeta^{2k} b, 3\zeta^{2k}, 0) \mid (a,b) \in \mathcal{C}_g,\ k \in \{0,1,2\}\right\} \subset \mathbf{C}^4.$$

Here $\zeta = e^{2\pi i/3}$, and $\mathcal{C}_g$ is the subset of the lattice $\mathbf{Z}^2$ given by

$$\mathcal{C}_g = \{(a,b) \in \mathbf{Z}^2 \mid |a| + |b| \leqslant 2g-2,\ a \equiv b \,(\mathrm{mod}\ 2)\}.$$

Then, the following is a partial analogue of (2.9).

Proposition 2.11. $\mathcal{E}_{g,d}^3 \subset \Xi_{g,d}^3$.

Proof. It is proved in [DX20, Prop. 5.7] that the set

$$\left\{(\sqrt{3}\zeta^{db} a, -\sqrt{-3}\zeta^{2db} b, 3\zeta^{2db}, 0) \mid (a,b) \in \mathcal{C}_g\right\} \tag{2.12}$$

is contained in $\Xi_{d,g}^3$. (Note that this set of eigenvalues differs slightly from that in [DX20, Prop. 5.7] because of Remark 2.3.) In fact, these eigenvalues simultaneously occur with the eigenvalue $+1$ of ε. The remaining eigenvalues are obtained using Lemma 2.5. ☐

The inclusion of Proposition 2.11 is conjectured to be equality, see [DX20, Conj. 7.3]. An analogue of (2.10) is essentially proved in [DX20] (see Proposition 2.23):

$$\dim V^3_{g,d}(\pm\sqrt{3}\zeta^k(2g-2), 0, 3\zeta^{2k}, 0) = 1 \tag{2.13}$$

where $k \in \{0,1,2\}$. Just as in the $N = 2$ case, the pairing (2.8) restricted to this 1-dimensional space is non-degenerate; Proposition 2.11 and property (2.13), with its non-degeneracy, are sufficient to define sutured instanton homology and prove an excision result in the case $N = 3$, parallel to the case $N = 2$, and this is explained in [DX20, §5.2].

In the case $N = 2$, Kronheimer and Mrowka prove a sutured decomposition result [KM10b, Prop. 7.11] using (2.9). This result implies that sutured instanton homology for taut sutured manifolds is nonzero, and leads to existence results for $U(2)$ representations. For $N = 3$, if equality in Proposition 2.11 holds, then similar arguments carry through. However, the inclusion of Proposition 2.11 by itself is not sufficient.

On the other hand, inspection of the arguments in [KM10b] shows that in the $N = 2$ case, equality of (2.9) is not necessary. The following weaker version of (2.9) suffices:

$$\Xi^2_{g,1} \cap (\mathbf{C} \times \{\pm 2\}) = \{(2ai^r, (-1)^r 2) \mid a \in \mathbf{Z},\ |a| \leqslant g-1,\ r \in \{0,1\}\}.$$

The same is true in the case $N = 3$: equality in Proposition 2.11 is not necessary, and the following, our main technical result, is a weaker version which suffices:

Theorem 2.14. *Let $d \in \mathbf{Z}$ be coprime to 3, $g \in \mathbf{Z}_{\geqslant 0}$, and $\zeta = e^{2\pi i/3}$. If $\lambda \in \Xi^3_{g,d}$ and $\lambda = (\lambda_1, \lambda_2, 3\zeta^j, 0)$ for some $j \in \mathbf{Z}$, then $\lambda \in \mathcal{E}^3_{g,d}$. Equivalently (by Proposition 2.11):*

$$\Xi^3_{g,d} \cap \left(\mathbf{C}^2 \times C_3 \times \{0\}\right) = \mathcal{E}^3_{g,d}$$

where $C_3 = \{3, 3\zeta, 3\zeta^2\}$ is the set of 3^{rd} roots of 27.

In the remainder of this section we explain the strategy to prove Theorem 2.14. Let $\mathcal{N}_g = \mathcal{N}^N_{g,d}$ be the moduli space of projectively flat $U(N)$-connections A on Σ_g with $\det(A) = A_0$, where A_0 is a fixed connection on a complex line bundle $L \to \Sigma_g$ of degree d. There is a natural isomorphism of rings (see Section 3):

$$H^*(\mathcal{N}_g; \mathbf{C}) = \mathbf{A}^N_g / I_g \tag{2.15}$$

where I_g is a homogeneous ideal in $\mathbf{A}^N_g$. Consider the extended ideal

$$I'_g := (\varepsilon^N - 1)I_g + \sum_{i=0}^{N-1} \varepsilon^i I_g \subset \mathbf{A}^N_g[\varepsilon].$$

Then, the relation ideal $J^N_{g,d}$ for $V^N_{g,d}$ from (2.7) is a deformation of the ideal I'_g. Concretely,

$$I'_g = \left(L(f) \mid f \in J^N_{g,d}\right) \subset \mathbf{A}^N_g[\varepsilon]$$

where $L(f)$ is the top degree homogeneous part of f. Here the degree of ε is set equal to 0. Furthermore, there is a complex vector space isomorphism

$$V_{g,d}^N \cong H^*(\mathcal{N}_g;\mathbf{C})[\varepsilon]/(\varepsilon^N - 1). \tag{2.16}$$

That is to say, the complex dimensions of the quotients of $\mathbf{A}_g^N[\varepsilon]$ by $J_{g,d}^N$ and I_g' are equal. These observations were first given by Muñoz in the case $N = 2$ [Muñ99]; the case for general N is similar, and discussed in [DX20].

Define the *simple type ideal* of $V_{g,d}^3$ as follows:

$$S_{g,d}^3 = \ker(\beta_2^3 - 27) \cap \ker(\beta_3) \cap \bigcap_{\substack{1\leqslant i\leqslant 2g\\ r=2,3}} \ker(\psi_r^i) \subset V_{g,d}^3. \tag{2.17}$$

The inclusion of Proposition 2.11 implies the following inequality:

$$\dim_{\mathbf{C}} S_{g,d}^3 \geqslant |\mathcal{E}_{g,d}^3| = 3(2g-1)^2. \tag{2.18}$$

Furthermore, if equality in (2.18) holds, then in it is straightforward to see that in fact there can be no other eigenvalues in $\Xi_{g,d}^3$ of the form $(\lambda_1, \lambda_2, 3\zeta^j, 0)$, and Theorem 2.14 follows. Thus our goal is to prove the inequality

$$\dim_{\mathbf{C}} S_{g,d}^3 \leqslant 3(2g-1)^2. \tag{2.19}$$

Define $\widetilde{J}_{g,d}^3$ to be the ideal of $\mathbf{A}_g^3[\varepsilon]$ generated by $J_{g,d}^3$ and $\beta_2^3 - 27$, β_3, ψ_r^i, $\varepsilon^3 - 1$. The pairing (2.8) satisfies $\langle ax, y\rangle = \langle x, ay\rangle$ for all $a \in V_{g,d}^N$. Thus there is an induced pairing

$$S_{g,d}^3 \otimes \mathbf{A}_g^3[\varepsilon]/\widetilde{J}_{g,d}^3 \to \mathbf{C}$$

Nondegeneracy of (2.8) implies the inequality

$$\dim_{\mathbf{C}} S_{g,d}^3 \leqslant \dim_{\mathbf{C}} \mathbf{A}_g^3[\varepsilon]/\widetilde{J}_{g,d}^3. \tag{2.20}$$

On the other hand, consider the ideal

$$\widetilde{I}_g := I_g + (\beta_2^3, \beta_3, \psi_2^i, \psi_3^i)_{1\leqslant i\leqslant 2g} \subset \mathbf{A}_g^3$$

and its extension $\widetilde{I}_g' := (\varepsilon^3 - 1)\widetilde{I}_g + \widetilde{I}_g + \varepsilon\widetilde{I}_g + \varepsilon^2\widetilde{I}_g$ inside $\mathbf{A}_g^3[\varepsilon]$. Since $\widetilde{J}_{g,d}^3$ is a deformation of the ideal $\widetilde{I}_g'$, it follows that there is an inequality

$$\dim_{\mathbf{C}} \mathbf{A}_g^3[\varepsilon]/\widetilde{J}_{g,d}^3 \leqslant \dim_{\mathbf{C}} \mathbf{A}_g^3[\varepsilon]/\widetilde{I}_g' = 3\dim_{\mathbf{C}} \mathbf{A}_g^3/\widetilde{I}_g. \tag{2.21}$$

Therefore, the following result, together with inequalities (2.20) and (2.21), proves the desired inequality (2.19), and hence proves Theorem 2.14.

Theorem 2.22. *For $g \geqslant 1$, $\dim_{\mathbf{C}} \mathbf{A}_g^3/\widetilde{I}_g \leqslant (2g-1)^2$.*

This theorem is proved in the next two sections, where the ring $H^*(\mathcal{N}_g)$ is studied. For reasons explained above (see also the end of this section), we call $\mathbf{A}_g^3/\widetilde{I}_g$ the *undeformed simple type quotient*.

We now show how (2.13) follows from [DX20] and Theorem 2.22.

Proposition 2.23. $\dim V_{g,d}^3(\pm\sqrt{3}\zeta^k(2g-2), 0, 3\zeta^{2k}, 0) = 1$ *for each* $k \in \{0,1,2\}$. *In particular, the generalized eigenspace for* $(\pm\sqrt{3}\zeta^k(2g-2), 0, 3\zeta^{2k}, 0)$ *agrees with the corresponding eigenspace.*

Proof. From our above discussion, Theorem 2.22 implies

$$\dim_{\mathbf{C}} \mathbf{A}_g^3[\varepsilon]/\widetilde{J}_{g,d}^3 = 3(2g-1)^2. \tag{2.24}$$

For $\lambda_0 \in \mathbf{C}$ and $\lambda \in \Xi_{g,d}^3$ write $V(\lambda_0, \lambda) = V_{g,d}^3(\lambda) \cap \ker(\varepsilon - \lambda_0)$. Then

$$V_{g,d}^3 = \bigoplus_{(\lambda_0,\lambda)\in\mathbf{C}\times\Xi_{g,d}^3} V(\lambda_0, \lambda).$$

Write Π for the projection from $V_{g,d}^3$ to $\mathbf{A}_g^3[\varepsilon]/\widetilde{J}_{g,d}^3$. Since $|\mathcal{E}_{g,d}^3| = 3(2g-1)^2$, for each $\lambda \in \mathcal{E}_{g,d}^3$ there is a *unique* $\lambda_0 \in \mathbf{C}$ such that $\Pi(V(\lambda_0, \lambda))$ is nonzero. For if this were not the case, the equality (2.24) would be violated. Let $\lambda = (\pm\sqrt{3}(2g-2), 0, 3, 0)$. In [DX20, §5.1] it is shown that $V(1, \lambda)$ is 1-dimensional. Consequently,

$$\dim \left(V_{g,d}^3(\lambda) \cap \ker(\varepsilon - 1)\right) = 1.$$

On the other hand, by the above remarks, it must be that $V_{g,d}^3(\lambda) \subset \ker(\varepsilon - 1)$. This proves the desired result in the case $k = 0$. The cases where $k \in \{1, 2\}$ then follow from the case $k = 0$ and Lemma 2.5. □

We conclude this section with some commentary on our terminology used for the subspace $S_{g,d}^3 \subset V_{g,d}^3$. First, for any oriented smooth 4-manifold X recall the definition

$$\mathbf{A}^3(X) = (\mathrm{Sym}^*(H_0(X) \otimes H_2(X)) \otimes \Lambda^* H_1(X))^{\otimes 2}$$

where complex coefficients are assumed. If X is closed and $b^+(X) > 1$, and $w \in H^2(X; \mathbf{Z})$, then there is an associated $U(3)$ Donaldson-type invariant

$$D_{X,w}^3 : \mathbf{A}^3(X) \to \mathbf{C},$$

and we now review the outline of its construction. Let $z = (z_{i_1} \cdots z_{i_k}) \otimes (z'_{j_1} \cdots z'_{j_l}) \in \mathbf{A}^3(X)$ where each z_{i_s} and z'_{j_s} in $H_i(X)$ for some $i \in \{0, 1, 2\}$. Consider the moduli space of $PU(3)$ instantons with energy κ on X, with bundle determined by w, and cut down by divisors representing $\mu_2(z_{i_1}), \ldots \mu_2(z_{i_k}), \mu_3(z'_{j_1}), \ldots, \mu_3(z'_{j_l})$. The energy κ is chosen so that the cut-down space has expected dimension 0 (if this is not possible, the invariant is zero). For a generic metric and perturbation, the cut-down moduli space is a compact 0-manifold, and $D_{X,w}^3(z)$ is the associated signed count. (In general, the blow-up trick of Morgan–Mrowka is also employed.) The following condition on pairs (X, w) refines the definition of $U(3)$ simple type given in the introduction.

Definition 2.25. Let X be a closed oriented 4-manifold with $b^+(X) > 1$ and $w \in H^2(X; \mathbf{Z})$. The pair (X, w) is called $U(3)$ *simple type* if

$$D^3_{X,w}((x^3_{(2)} - 27)z) = 0, \quad D^3_{X,w}(x_{(3)}z) = 0, \quad D^3_{X,w}(\delta z) = 0 \tag{2.26}$$

for any $z \in \mathbf{A}^3(X)$ and any $\delta \in \Lambda^* H_1(X) \otimes \Lambda^* H_1(X) \subset \mathbf{A}^3(X)$. We say X is $U(3)$ *simple type* if (X, w) is $U(3)$ simple type for all $w \in H^2(X; \mathbf{Z})$.

Let (X, w) be a pair of a closed, smooth, oriented 4-manifold and a 2-cycle w, with $b_1(X) = 0$ and $b^+(X) > 1$, which is also $U(3)$ simple type. Suppose further that $\Sigma \subset X$ is an embedded surface of genus g in X such that $\Sigma \cdot \Sigma = 0$ and $d := \Sigma \cdot w$ is coprime to 3. Removing a regular neighborhood of Σ from (X, w) produces a pair (X°, w°) with boundary $(S^1 \times \Sigma_g, \gamma_d)$. In particular, for any $z \in \mathbf{A}^3(X^\circ)$ there are relative invariants

$$D^3_{X^\circ, w^\circ}(z) \in V^3_{g,d}. \tag{2.27}$$

The proof of (2.11) from [DX20] produces eigenvectors with eigenvalues in $\mathcal{E}^3_{g,d}$ using such relative invariants (see also proof of Theorem 6.15). A gluing formula expresses invariants of (X, w) in terms of relative invariants, using the pairing (2.8):

$$D^3_{X,w}(zz') = \langle D^3_{X^\circ, w^\circ}(z), z'\mathbf{1}\rangle, \tag{2.28}$$

where $z' \in \mathbf{A}^3_{g,d}$, which also induces an element of $\mathbf{A}^3(X)$. The gluing formula (2.28), the simple type condition (2.26), and the non-degeneracy of the pairing (2.8) imply that

$$D^3_{X^\circ, w^\circ}(z) \in S^3_{g,d}. \tag{2.29}$$

A consequence of Theorem 2.22 is

$$\dim_{\mathbf{C}} S^3_{g,d} = 3(2g-1)^2,$$

which implies that the simple type ideal $S^3_{g,d}$ is in fact spanned by relative invariants coming from simple type 4-manifolds.

3 Mumford relations and their duals

As in the previous section, denote by $\mathcal{N}_g = \mathcal{N}^N_{g,d}$ the moduli space of projectively flat $U(N)$-connections A on a Riemann surface Σ_g of genus g with $\det(A) = A_0$, where A_0 is a fixed connection on a line bundle $L \to \Sigma_g$ of degree $d \in \mathbf{Z}$. Assume as before that d is coprime to N. Then $\mathcal{N}_g$ is a smooth manifold of dimension $(N^2-1)(2g-2)$. By the Narasimhan–Seshadri correspondence, $\mathcal{N}_g$ may be identified with the moduli space of rank N stable holomorphic bundles over Σ_g with fixed determinant of degree d.

There is a universal $U(N)$-bundle $U \to \mathcal{N}_g \times \Sigma_g$. This bundle is not unique, as tensoring it by any line bundle pulled back from $\mathcal{N}_g$ gives another such choice. However,

$$\mathbf{P} := U \otimes \det(U)^{-1/N}$$

defines an element in the rational K-theory of $\mathcal{N}_g \times \Sigma_g$, which is independent of the choice of the universal bundle U. We define cohomology classes

$$\alpha_r \in H^{2r-2}(\mathcal{N}_g), \qquad \psi_r^i \in H^{2r-1}(\mathcal{N}_g), \qquad \beta_r \in H^{2r}(\mathcal{N}_g) \tag{3.1}$$

using the Künneth decomposition of the Chern class $c_r(\mathbf{P}) \in H^*(\mathcal{N}_g) \otimes H^*(\Sigma_g)$:

$$c_r(\mathbf{P}) = \alpha_r \otimes \sigma + \sum_{i=1}^{2g} \psi_r^i \otimes \xi_i + \beta_r \otimes 1 \qquad (2 \leqslant r \leqslant N). \tag{3.2}$$

Note $c_1(\mathbf{P}) = 0$. All cohomology groups are defined over $\mathbf{C}$, unless otherwise mentioned. (However, everything in this section can be done over $\mathbf{Q}$.) Recall that a symplectic integral basis of $\{\eta_i\}_{i=1}^{2g}$ for $H_1(\Sigma_g)$ was fixed earlier. In (3.2), $\{\xi_i\}_{i=1}^{2g}$ is the integral basis of $H^1(\Sigma_g)$ satisfying $\xi_i(\eta_j) = \delta_{ij}$, and σ is the integral generator of $H^2(\Sigma_g)$ given by the orientation. In particular, for $1 \leqslant i \leqslant g$, we have $\xi_i\xi_{g+i} = \sigma$ and $\xi_i\xi_j = 0$ when $j \neq g+i$. The following is a reformulation of a result due to Atiyah and Bott [AB83, Thm. 9.11] (see also [DX20, Prop. 3.14]).

Proposition 3.3. *The cohomology ring $H^*(\mathcal{N}_g)$ is generated by the elements $\alpha_r, \beta_r, \psi_r^i$ where $2 \leqslant r \leqslant N$, $1 \leqslant i \leqslant 2g$.*

This result induces the isomorphism (2.15) mentioned earlier.

We next turn to relations for these generators. Let J_g be the Jacobian torus of Σ_g, viewed as the moduli space of flat $U(1)$-connections on Σ_g, or equivalently, the moduli space of holomorphic line bundles of degree zero. Let

$$V \to \mathcal{N}_g \times J_g \times \Sigma_g.$$

be defined as the tensor product of the pullback of U with the pullback of the Poincaré bundle over $J_g \times \Sigma_g$. We have

$$c_1(V) = d \cdot 1 \otimes 1 \otimes \sigma + \sum_{i=1}^{2g} 1 \otimes d_i \otimes \xi_i + x \otimes 1 \tag{3.4}$$

where $x \in H^2(\mathcal{N}_g \times J_g)$, and $d_i \in H^1(J_g)$ generate $H^*(J_g)$ as an exterior algebra. Consider the projection $f : \mathcal{N}_g \times J_g \times \Sigma_g \to \mathcal{N}_g \times J_g$. The Grothendieck–Riemann–Roch formula expresses $c_i(f_!V)$ in terms of the generators (3.1) and elements of $H^*(J_g)$. Now assume

$$d = 2N(g-1) + d', \qquad 1 \leqslant d' < N. \tag{3.5}$$

Throughout this subsection, d' is fixed, and g is a positive integer. As a consequence of stability and Serre duality, $H^1(\Sigma_g; \mathcal{E} \otimes \mathcal{L}) = 0$ for any stable rank N bundle $\mathcal{E}$ and degree zero holomorphic line bundle $\mathcal{L}$ over Σ_g. Therefore $f_!V$ is an honest vector bundle over $\mathcal{N}_g \times J_g$, whose rank can be computed using Riemann-Roch. Consequently, we have

$$c_i(f_!V) = 0 \quad \text{if} \quad i > \mathrm{rk}(f_!V) = N(g-1) + d'. \tag{3.6}$$

Taking slant products of the Chern classes (3.6) with elements in $H^*(J_g)$ thus yields relations for the generators (3.1). We call these *Mumford relations*, following the discussion in [AB83]. In the case $N = 2$, the Mumford relations were shown to be a complete set of relations for the ring $H^*(\mathcal{N}_g)$ by Kirwan [Kir92].

When $N > 2$, the Mumford relations do not give a complete set of relations for $H^*(\mathcal{N}_g)$. Following [Ear97], we consider a line bundle $L \to \Sigma_g$ of degree $4(g-1)+1$. Let $\phi : \mathcal{N}_g \times J_g \times \Sigma_g \to \Sigma_g$ be projection. Define the "dual" universal bundle

$$\overline{V} := V^* \otimes \phi^* L.$$

Then under assumption (3.5), a similar argument using stability and Serre duality implies that $f_! \overline{V}$ is an honest vector bundle. We then obtain

$$c_i(f_! \overline{V}) = 0 \quad \text{if} \quad i > \mathrm{rk}(f_! \overline{V}) = Ng - d'. \tag{3.7}$$

Again, taking slant products of the classes (3.6) with elements in $H^*(J_g)$ yields relations for the generators (3.1). We call these *dual Mumford relations*, following Earl. In the case $N = 3$, the work of Earl [Ear97] implies that the Mumford relations and the dual Mumford relations form a complete set of relations for $H^*(\mathcal{N}_g)$.

In the following, we use Grothendieck–Riemann–Roch to compute the Chern classes of $f_! V$, $f_! \overline{V}$ and then use (3.6) and (3.7) to obtain relations in the cohomology ring $H^*(\mathcal{N}_g)$. For this purpose, we may assume x in (3.4) is zero, by tensoring V by a formal line bundle over $\mathcal{N}_g \times J_g$ whose first Chern class is $-x/N$. Following an observation of Zagier [Zag95, p.22], this assumption does not affect (3.6) and (3.7).

When $N > 3$, the Mumford relations and dual Mumford relations are not complete, and more relations are necessary. A complete set of relations for general N was given by Earl and Kirwan [EK04]. As our focus is the case $N = 3$, we will only consider the Mumford and dual Mumford relations. The particular elements we consider are

$$\zeta_m^{g,k} := (-N)^k c_{m+k}(f_! V)/D_k, \qquad \overline{\zeta}_m^{g,k} := (-N)^k c_{m+k}(f_! \overline{V})/D_k \tag{3.8}$$

where $D_k \in H_{2k}(J_g)$ has pairing 1 with $d_1 d_{g+1} d_2 d_{g+2} \cdots d_k d_{g+k} \in H^{2k}(J_g)$ and trivial pairing with other exterior products of d_i. More precisely, we consider these classes in terms of the generators (3.1) as derived from Grothendieck–Riemann–Roch. Thus

$$\zeta_m^{g,k}, \ \ \overline{\zeta}_m^{g,k} \ \ \in \mathbf{A}_g^N = \mathbf{C}[\alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N] \otimes \Lambda^*(\psi_r^i)$$

By our discussion thus far, the cohomology ring for $\mathcal{N}_g$ may be written as in (2.15),

$$H^*(\mathcal{N}_g) = \mathbf{A}_g^N / I_g,$$

and using (3.6), (3.7), the ideal of relations I_g contains the following elements:

$$\zeta_m^{g,k} \in I_g \quad \text{if} \quad m > N(g-1) - k + d', \tag{3.9}$$

$$\overline{\zeta}_m^{g,k} \in I_g \quad \text{if} \quad m > Ng - k - d'. \tag{3.10}$$

We now study these relations after modding out by the classes ψ_r^i. In the computations below, there will frequently appear two constants:

$$\mathsf{c}_{N,d'} := 1 - \frac{d'}{N}, \qquad \bar{\mathsf{c}}_{N,d'} := 1 - \mathsf{c}_{N,d'} = \frac{d'}{N}. \tag{3.11}$$

We first obtain an expression for the generating functions of the polynomials $\zeta_m^{g,k}$ (with respect to the index m). Below, the notation "$\equiv_\psi$" means congruence modulo the ideal $(\psi_r^i)_{2 \leqslant r \leqslant N, 1 \leqslant i \leqslant 2g}$. By convention, we also set $\beta_1 = \alpha_1 = \alpha_0 = 0$ and $\beta_0 = 1$.

Proposition 3.12. *The generating series $F_{g,k}(t) := \sum_{m=0}^\infty \zeta_m^{g,k} t^m$ mod (ψ_r^i) is given by:*

$$F_{g,k}(t) \equiv_\psi \left(\sum_{i=0}^N \beta_i t^i\right)^{g-k-\mathsf{c}_{N,d'}} \left(\sum_{i=0}^N (1 - \frac{i}{N}) \beta_i t^i\right)^k G(t)$$

where the power series $G(t) \in \mathbf{Q}[\alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N][\![t]\!]$ is defined by

$$G(t) = \exp\left(\sum_{i=1}^N \alpha_i \frac{\partial}{\partial \beta_i} \left(\sum_{n=1}^\infty - \frac{(-t)^n p_{n+1}}{n(n+1)}\right)\right). \tag{3.13}$$

The notation p_n refers to the n^{th} power symmetric function, viewed as a function of the elementary symmetric functions. More explicitly, recall that given variables $x_1, x_2, \ldots$ there are the elementary symmetric functions e_n and the power symmetic functions p_n:

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \qquad\qquad p_n = \sum_i x_i^n$$

It is a basic result that p_n can be written as a function in the elementary symmetric functions: $p_n = p_n(e_1, e_2, \ldots)$. An explicit relationship is given by

$$-\sum_{n=1}^\infty \frac{(-t)^n p_n}{n} = \log\left(\sum_{n=0}^\infty e_n t^n\right). \tag{3.14}$$

In the formula of Proposition 3.12, $\partial p_n / \partial \beta_i$ should be interpreted by identifying the β_i with e_i for $i = 0$ and $2 \leqslant i \leqslant N$, and setting the other $e_i = 0$. Explicitly,

$$\frac{\partial p_n}{\partial \beta_i} = \frac{\partial}{\partial e_i} p_n(e_1, e_2, \ldots)\Big|_{e_i = \beta_i}$$

Proof of Proposition 3.12. We adapt the computation of Zagier [Zag95, §6], which is for the case $N = 2$. Grothendieck–Riemann–Roch says

$$\mathrm{ch}(f_! V) = \left(\mathrm{ch}(V) \mathrm{td}(\Sigma_g)\right) / [\Sigma_g] \tag{3.15}$$

where $\mathrm{td}(\Sigma_g) = 1 - (g-1)\sigma$ is the Todd class of Σ_g. We have

$$\mathrm{ch}(V) = \mathrm{ch}(\mathbf{P})\mathrm{ch}(\det(V)^{1/N})$$

where $\mathrm{ch}(\mathbf{P})$ is interpreted via the natural map $H^*(\mathcal{N}_g \times \Sigma_g) \to H^*(\mathcal{N}_g \times J_g \times \Sigma_g)$ induced by projection. Using (3.4) (and recalling $x = 0$) we obtain

$$\mathrm{ch}(\det(V)^{1/N}) = 1 + \frac{1}{N} d \otimes 1 \otimes \sigma + \frac{1}{N}\sum_{i=1}^{2g} 1 \otimes d_i \otimes \xi_i - \frac{1}{N^2} 1 \otimes A \otimes \sigma$$

where $A = \sum_{i=1}^{g} d_i d_{g+i} \in H^2(J_g)$. We next give an expression for ch($\mathbf{P}$). Let γ_i (resp. δ_i), where $1 \leqslant i \leqslant N$, be formal degree 2 classes such that the i^{th} elementary symmetric polynomial in the γ_i (resp. δ_i) is equal to $c_i(\mathbf{P})$ (resp. β_i). Then

$$\mathrm{ch}(\mathbf{P}) \quad = \quad \sum_{i=1}^{N} e^{\gamma_i} \quad = \quad \sum a_{r_1,\ldots,r_N} c_1(\mathbf{P})^{r_1} \cdots c_N(\mathbf{P})^{r_N}$$

for some constants $a_{r_1,\ldots,r_N}$. A direct computation from (3.2) gives

$$c_1(\mathbf{P})^{r_1} \cdots c_N(\mathbf{P})^{r_N} \;\equiv_\psi\; \beta_1^{r_1} \cdots \beta_N^{r_N} \otimes 1 \otimes 1 + \sum_{i=0}^{N} \alpha_i \frac{\partial}{\partial \beta_i}(\beta_1^{r_1} \cdots \beta_N^{r_N}) \otimes 1 \otimes \sigma.$$

Together with the identity $\sum_{i=1}^{N} e^{\delta_i} = \sum a_{r_1,\ldots,r_N} \beta_1^{r_1} \cdots \beta_N^{r_N}$, these relations yield

$$\mathrm{ch}(\mathbf{P}) \quad \equiv_\psi \quad \sum_{i=1}^{N} e^{\delta_i} \otimes 1 \otimes 1 + \sum_{i,j=1}^{N} \alpha_i \frac{\partial}{\partial \beta_i}(e^{\delta_j}) \otimes 1 \otimes \sigma$$

With these observations in hand, we evaluate (3.15) to be

$$\mathrm{ch}(f_! V) \quad \equiv_\psi \quad (\frac{d}{N} - (g-1) - \frac{1}{N^2} A) \sum_{i=1}^{N} e^{\delta_i} \otimes 1 + \sum_{i,j=1}^{N} \alpha_i \frac{\partial}{\partial \beta_i} e^{\delta_j} \otimes 1 \tag{3.16}$$

To determine the Chern classes from this expression, we use the following [Zag95, Lemma 1]: for any vector bundle E over a space X, we have

$$\log c(E) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} s_n \qquad \Longleftrightarrow \qquad \mathrm{ch}(E) = \mathrm{rk}(E) + \sum_{n=1}^{\infty} \frac{s_n}{n!} \tag{3.17}$$

where $s_n = s_n(E) \in H^{2n}(X)$. For $E = f_! V$, using (3.16) we compute

$$s_n \;\equiv_\psi\; (\frac{d}{N} - (g-1)) p_n - \frac{n}{N^2} A p_{n-1} + \sum_{i=1}^{N} \frac{1}{n+1} \alpha_i \frac{\partial}{\partial \beta_i} p_{n+1}$$

where p_n is the n^{th} power symmetric function in $\delta_1, \ldots, \delta_N$. From (3.17), we obtain

$$c(f_!V)_t \equiv_\psi \left(\sum_{i=0}^N \beta_i t^i\right)^{\frac{d}{N}-(g-1)} \exp\left(-\frac{At}{N^2}\sum_{n=1}^\infty (-t)^{n-1}p_{n-1}\right) G(t) \tag{3.18}$$

Here we use the notation $c(E)_t = \sum_{i=0}^\infty c_i(E)t^i$ for the power series associated to the total Chern class of E. Using $A^r/D_k = r!\delta_{rk}$, we obtain an expression for the slant product:

$$c(f_!V)_t/D_k \equiv_\psi \left(\sum_{i=0}^N \beta_i t^i\right)^{\frac{d}{N}-(g-1)} \left(\frac{-t}{N^2}\sum_{n=1}^\infty (-t)^{n-1}p_{n-1}\right)^k G(t)$$

Taking the derivative of relation (3.14) (with $e_k = \beta_k$) gives

$$\sum_{n=1}^\infty (-t)^{n-1}p_n = \sum_{i=0}^N i\beta_i t^{i-1} / \sum_{i=0}^N \beta_i t^i.$$

Noting $p_0 = N$, we then have the relation

$$\sum_{n=1}^\infty (-t)^{n-1}p_{n-1} = N - \sum_{i=0}^N i\beta_i t^i / \sum_{i=0}^N \beta_i t^i = \sum_{i=0}^N (N-i)\beta_i t^i / \sum_{i=0}^N \beta_i t^i.$$

Substituting this last expression into (3.18), and using $t^k F_{g,k}(t) = (-N)^k c(f_!V)_t/D_k$, as determined by (3.8), gives the result. □

Remark 3.19. Setting all ψ_r^i equal to zero simplifies this computation considerably. Explicit formulas can of course be obtained without this simplification; see [Ear97, Kir92] for computations along these lines (using different generators).

From Proposition 3.12 we derive a recursive relation for the $\zeta_m^{g,k}$.

Proposition 3.20. *The polynomials* $\zeta_m^{g,k} \bmod (\psi_r^i)$ *in the ring* $\mathbf{C}[\alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N]$ *are determined recursively, for fixed* g*,* k*, as follows (with* $\zeta_0^{g,k} = 1$ *and* $\zeta_m^{g,k} = 0$ *for* $m < 0$*):*

$$N(m+1)\zeta_{m+1}^{g,k} \equiv_\psi -\sum_{i,j=0}^N (N-j)\alpha_i\beta_j\zeta_{m-i-j+2}^{g,k} + \tag{3.21}$$

$$\sum_{\substack{i,j=0\\(i,j)\neq(0,0)}}^N ((g-k-\mathsf{c}_{N,d'})i(N-j) - (N-i)(m-i-j+1) + kj(N-j))\beta_i\beta_j\zeta_{m-i-j+1}^{g,k}$$

Proof. First consider the series $G(t)$ from (3.13). We compute

$$\frac{G'(t)}{G(t)} = \sum_{i=0}^N t^{-2}\alpha_i \frac{\partial}{\partial\beta_i}\left(\sum_{n=1}^\infty \frac{(-1)^{n+1}p_{n+1}t^{n+1}}{n+1}\right)$$

$$= -\sum_{i=0}^N t^{-2}\alpha_i\frac{\partial}{\partial\beta_i}\left(\log\sum_{j=0}^N \beta_j t^j\right) = -\sum_{i=0}^N \alpha_i t^{i-2} / \sum_{j=0}^N \beta_j t^j.$$

In the second equality, we have again used (3.14) with $e_k = \beta_k$. Next, we compute

$$\frac{F'_{g,k}(t)}{F_{g,k}(t)} = (g - k - \mathsf{c}_{N,d'}) \left(\sum_{i=0}^{N} i\beta_i t^{i-1} \right) \left(\sum_{i=0}^{N} \beta_i t^i \right)^{-1}$$

$$+k \left(\sum_{i=0}^{N} i(1 - \frac{i}{N})\beta_i t^{i-1} \right) \left(\sum_{i=0}^{N} (1 - \frac{i}{N})\beta_i t^i \right)^{-1} - \left(\sum_{i=0}^{N} \alpha_i t^{i-2} \right) \left(\sum_{i=0}^{N} \beta_i t^i \right)^{-1}.$$

Multiply both sides by $(\sum_{i=0}^{N} \beta_i t^i)(\sum_{i=0}^{N} (N-i)\beta_i t^i) F_{g,k}(t)$. Then, using

$$F_{g,k}(t) = \sum_{m=0}^{\infty} \zeta_m^{g,k} t^m, \qquad F'_{g,k}(t) = \sum_{m=0}^{\infty} m\zeta_m^{g,k} t^{m-1},$$

the desired recursion follows by extracting the coefficient of t^m from each side. $\square$

Many other recursions may be extracted from Proposition 3.12. For example:

Proposition 3.22. *The polynomials* $\zeta_m^{g,k} \bmod (\psi_r^i)$ *in* $\mathbf{C}[\alpha_2, \ldots, \alpha_N, \beta_2, \ldots, \beta_N]$ *satisfy:*

$$\zeta_m^{g+1,k} \equiv_\psi \sum_{i=0}^{N} \beta_i \zeta_{m-i}^{g,k} \tag{3.23}$$

$$\sum_{i=0}^{N} \beta_i \zeta_{m-i}^{g,k+1} \equiv_\psi \sum_{i=0}^{N} (1 - i/N)\, \beta_i \zeta_{m-i}^{g,k} \tag{3.24}$$

Proof. The first relation follows using $F_{g+1,k}(t) = (\sum_{i=0}^{N} \beta_i t^i) F_{g,k}(t)$, and the second relation follows from $(\sum_{i=0}^{N} \beta_i t^i) F_{g,k+1}(t) = (\sum_{i=0}^{N} (1 - i/N)\beta_i t^i) F_{g,k}(t)$. $\square$

The case of the dual Mumford elements $\overline{\zeta}_m^{g,k}$ is similar. In fact, if one makes the following changes to the expression for $F_{g,k}(t)$ in Proposition 3.12:

$$\alpha_i \mapsto \overline{\alpha}_i := (-1)^i \alpha_i, \qquad \beta_i \mapsto \overline{\beta}_i := (-1)^i \beta_i, \qquad \mathsf{c}_{N,d'} \mapsto \overline{\mathsf{c}}_{N,d'}, \tag{3.25}$$

then one obtains the generating function for $\overline{\zeta}_m^{g,k}$. For example, recursion (3.21) becomes

$$N(m+1)\overline{\zeta}_{m+1}^{g,k} \equiv_\psi - \sum_{i,j=0}^{N} (N-j)\overline{\alpha}_i \overline{\beta}_j \overline{\zeta}_{m-i-j+2}^{g,k} +$$

$$\sum_{\substack{i,j=0 \\ (i,j)\neq(0,0)}}^{N} ((g - k - \overline{\mathsf{c}}_{N,d'})i(N-j) - (N-i)(m-i-j+1) + kj(N-j))\overline{\beta}_i \overline{\beta}_j \overline{\zeta}_{m-i-j+1}^{g,k}$$

with the same initial conditions: $\overline{\zeta}_0^{g,k} = 1$ and $\overline{\zeta}_m^{g,k} = 0$ for $m < 0$.

In the remainder of this subsection, we prove two lemmas that will be used later to understand the ideal of relations in the case of $N = 3$.

Lemma 3.26. *Suppose $N \geqslant 3$. For any integers g, k, m with $k \neq N/2 - 1$ we have:*

$$\beta_2^2 \zeta_{m-1}^{g,k} \in (\zeta_{m+1}^{g,k}, \zeta_{m+2}^{g,k}, \zeta_{m+3}^{g,k}, \alpha_a, \beta_2^3, \beta_b, \psi_r^i), \tag{3.27}$$

$$\beta_2^2 \overline{\zeta}_{m-1}^{g,k} \in (\overline{\zeta}_{m+1}^{g,k}, \overline{\zeta}_{m+2}^{g,k}, \overline{\zeta}_{m+3}^{g,k}, \alpha_a, \beta_2^3, \beta_b, \psi_r^i), \tag{3.28}$$

where the indices range over $4 \leqslant a \leqslant N$, $3 \leqslant b \leqslant N$, $2 \leqslant r \leqslant N$, $1 \leqslant i \leqslant 2g$.

Proof. In this proof we write "$\equiv$" to mean congruent modulo $(\beta_2^3, \beta_b, \psi_r^i)$ with $3 \leqslant b \leqslant N$, $2 \leqslant r \leqslant N$, $1 \leqslant i \leqslant 2g$. We prove (3.27), the case of (3.28) being similar. First note that we can write (3.21) as follows:

$$(m+1)\zeta_{m+1}^{g,k} \equiv -\sum_{i=2}^{N} \alpha_i \zeta_{m-i+2}^{g,k} - q\sum_{i=2}^{N} \alpha_i \beta_2 \zeta_{m-i}^{g,k} + r_m^{g,k} \beta_2 \zeta_{m-1}^{g,k} + s_m^g \beta_2^2 \zeta_{m-3}^{g,k}. \tag{3.29}$$

The constants here are given by $q = (N-2)/N$ and

$$r_m^{g,k} = 2g - \frac{4}{N}k - 2\mathsf{c}_{N,d'} + 2(m-1)\left(\frac{1}{N} - 1\right),$$

$$s_m^g = \left(\frac{N-2}{N}\right)(2g - m - 2\mathsf{c}_{N,d'} + 3).$$

Now multiply (3.29) by β_2 to obtain the following:

$$(m+1)\beta_2\zeta_{m+1}^{g,k} \equiv -\sum_{i=2}^{N} \alpha_i \beta_2 \zeta_{m-i+2}^{g,k} - q\sum_{i=2}^{N} \alpha_i \beta_2^2 \zeta_{m-i}^{g,k} + r_m^{g,k} \beta_2^2 \zeta_{m-1}^{g,k}. \tag{3.30}$$

Multiplying once more by β_2 gives

$$(m+1)\beta_2^2\zeta_{m+1}^{g,k} \equiv -\sum_{i=2}^{N} \alpha_i \beta_2^2 \zeta_{m-i+2}^{g,k}. \tag{3.31}$$

We can use (3.31) to rewrite the middle term on the right side of (3.30), yielding:

$$(m+1)\beta_2\zeta_{m+1}^{g,k} \equiv -\sum_{i=2}^{N} \alpha_i \beta_2 \zeta_{m-i+2}^{g,k} + q(m+1)\beta_2^2 \zeta_{m-1}^{g,k} + r_m^{g,k} \beta_2^2 \zeta_{m-1}^{g,k}. \tag{3.32}$$

The first term on the right side of (3.32) can be rewritten using (3.29):

$$\begin{aligned} -\sum_{i=2}^{N} \alpha_i \beta_2 \zeta_{m-i+2}^{g,k} \equiv{} & q^{-1}(m+3)\zeta_{m+3}^{g,k} + q^{-1}\sum_{i=2}^{N} \alpha_i \zeta_{m-i+4}^{g,k} \\ & - q^{-1} r_{m+2}^{g,k} \beta_2 \zeta_{m+1}^{g,k} - q^{-1} s_{m+2}^g \beta_2^2 \zeta_{m-1}^{g,k}. \end{aligned}$$

Note $q \neq 0$ since $N \geqslant 3$. Substituting this into (3.32) we obtain

$$(m+1)\beta_2\zeta^{g,k}_{m+1} \equiv q^{-1}(m+3)\zeta^{g,k}_{m+3} + q^{-1}\sum_{i=2}^{N}\alpha_i\zeta^{g,k}_{m-i+4} - q^{-1}r^{g,k}_{m+2}\beta_2\zeta^{g,k}_{m+1}$$
$$- q^{-1}s^{g}_{m+2}\beta_2^2\zeta^{g,k}_{m-1} + q(m+1)\beta_2^2\zeta^{g,k}_{m-1} + r^{g,k}_{m}\beta_2^2\zeta^{g,k}_{m-1}.$$

Rearranging, we obtain the following expression:

$$\left(q^{-1}s^{g}_{m+2} - q(m+1) - r^{g,k}_{m}\right)\beta_2^2\zeta^{g,k}_{m-1} \equiv$$
$$q^{-1}(m+3)\zeta^{g,k}_{m+3} + q^{-1}\sum_{i=2}^{N}\alpha_i\zeta^{g,k}_{m-i+4} - q^{-1}r^{g,k}_{m+2}\beta_2\zeta^{g,k}_{m+1} - (m+1)\beta_2\zeta^{g,k}_{m+1}.$$

The constant on the left side is equal to $(4k+4-2N)/N$, which is non-zero under the assumption of the proposition statement. Inspection of the right side of this last expression proves that $\beta_2^2\zeta^{g,k}_{m-1}$ is in the ideal claimed. □

Lemma 3.33. *Suppose $N \geqslant 3$. For any integers g, k, m we have the following:*

$$\beta_2\zeta^{g,k+1}_{m-2} \in (\zeta^{g,k+1}_{m}, \zeta^{g,k}_{m}, \beta_2^2, \beta_b, \psi^i_r), \tag{3.34}$$

$$\beta_2\overline{\zeta}^{g,k+1}_{m-2} \in (\overline{\zeta}^{g,k+1}_{m}, \overline{\zeta}^{g,k}_{m}, \beta_2^2, \beta_b, \psi^i_r), \tag{3.35}$$

where the indices range over $3 \leqslant b \leqslant N$, $2 \leqslant r \leqslant N$, $1 \leqslant i \leqslant 2g$.

Proof. We prove (3.34), the case (3.35) being similar. Consider relation (3.24):

$$\zeta^{g,k+1}_{m} + \beta_2\zeta^{g,k+1}_{m-2} \equiv \zeta^{g,k}_{m} + \frac{N-2}{N}\beta_2\zeta^{g,k}_{m-2}$$

where in this proof "≡" means congruent modulo the ideal $(\beta_2^2, \beta_b, \psi^i_r)$ where $b \geqslant 3$. Multiply this expression by β_2 and shift subscripts to obtain

$$\beta_2\zeta^{g,k+1}_{m-2} \equiv \beta_2\zeta^{g,k}_{m-2}.$$

These two expressions yield the following, which implies the result:

$$\beta_2\zeta^{g,k+1}_{m-2} \equiv \frac{N}{2}\left(\zeta^{g,k}_{m} - \zeta^{g,k+1}_{m}\right).$$ □

4 The $N = 3$ undeformed simple type quotient

We now specialize to the case $N = 3$ and focus on a quotient of the cohomology ring of the moduli space $\mathcal{N}_g = \mathcal{N}^N_{g,d}$. To be more specific, fix the choice of d' in (3.5) to be $d' = 1$. There is no loss in generality in making this choice, as the moduli space for $d' = 2$ may be identified with that for $d' = 1$ by the map which sends a stable rank 3 bundle to its conjugate. We take the quotient of $H^*(\mathcal{N}_g)$ by the curve classes ψ_r^i and the cohomology classes β_2^3, β_3 given by the point classes. This quotient is a cyclic module over the ring $\mathbf{A}_g^3$:

$$H^*(\mathcal{N}_g)/(\beta_2^3, \beta_3, \psi_2^i, \psi_3^i)_{1\leqslant i\leqslant 2g} = \mathbf{A}_g^3/\widetilde{I}_g \tag{4.1}$$

where $\widetilde{I}_g$ is the ideal $I_g + (\beta_2^3, \beta_3, \psi_2^i, \psi_3^i)_{1\leqslant i\leqslant 2g}$. This quotient was introduced in Section 2, where it was called the undeformed simple type quotient. Our main goal of this subsection is to prove Theorem 2.22, which we restate here:

$$\dim_{\mathbf{C}} \mathbf{A}_g^3/\widetilde{I}_g \leqslant (2g-1)^2 \qquad (g \geqslant 1).$$

We continue to work with coefficients in $\mathbf{C}$, following our convention as set in the previous sections, although everything here works over $\mathbf{Q}$.

We use the graded reverse lexicographic monomial ordering when dealing with polynomials in $\alpha_2, \alpha_3, \beta_2$, where the degrees of these elements are respectively $2, 4, 4$. In particular,

$$\alpha_2^i\alpha_3^j\beta_2^k > \alpha_2^{i'}\alpha_3^{j'}\beta_2^{k'}$$

if either $2i+4j+4k > 2i'+4j'+4k'$, or $2i+4j+4k = 2i'+4j'+4k'$ and the right-most nonzero entry of $(i-i', j-j', k-k')$ is negative. When taking leading terms below, it is always with respect to this monomial ordering. The leading term of a polynomial p in the variables $\alpha_2, \alpha_3, \beta_2$ with this convention is denoted $\mathrm{LT}(p)$.

We start with a simpler variation of (4.1) where we take the quotient of $H^*(\mathcal{N}_g)$ by the curve classes ψ_r^i and the point classes β_r with $r = 2, 3$ and $1 \leqslant i \leqslant 2g$. Modulo the curve and the point classes, the power series $F_{g,k}(t)$ or Proposition 3.12 is equal to $G(t)$, which is independent of g and k. In fact, modulo the point classes, $G(t)$ is equal to $\exp(-\alpha_2 t - \alpha_3 t^2/2)$. Motivated by this, let $\zeta_n, \overline{\zeta}_n \in \mathbf{C}[\alpha_2, \alpha_3]$ be defined by

$$\sum_{n=0}^{\infty} \zeta_n t^n = \exp(\alpha_2 t + \alpha_3 \frac{t^2}{2}), \qquad \sum_{n=0}^{\infty} \overline{\zeta}_n t^n = \exp(\alpha_2 t - \alpha_3 \frac{t^2}{2}).$$

More explicitly, we have the expressions

$$\zeta_n = \sum_{0\leqslant j\leqslant n/2} \frac{1}{(n-2j)!j!2^j}\alpha_2^{n-2j}\alpha_3^j, \qquad \overline{\zeta}_n = \sum_{0\leqslant j\leqslant n/2} \frac{(-1)^j}{(n-2j)!j!2^j}\alpha_2^{n-2j}\alpha_3^j.$$

These polynomials satisfy the recursive relations

$$m\zeta_m = \alpha_2\zeta_{m-1} + \alpha_3\zeta_{m-2}, \qquad m\overline{\zeta}_m = \alpha_2\overline{\zeta}_{m-1} - \alpha_3\overline{\zeta}_{m-2}, \tag{4.2}$$

in a similar way that $\zeta_m^{g,k}$ satisfy the relations in Proposition 3.20.

Proposition 4.3. *The leading term ideal of the ideal $I_n^0 := (\zeta_n, \zeta_{n+1}, \overline{\zeta}_{n+1}, \overline{\zeta}_{n+2})$ in the ring $\mathbf{C}[\alpha_2, \alpha_3]$ includes the following monomials:*

$$\{\alpha_2^i\alpha_3^j \mid 2i + 3j \geqslant 2n\}. \tag{4.4}$$

In fact, it will be a consequence of our proof of Theorem 2.22 that (4.4) is a generating set for the leading term ideal of I_n^0.

Proof. Define σ_n and $\overline{\sigma}_n$ respectively as $(\zeta_n + \overline{\zeta}_n)/2$ and $(\zeta_n - \overline{\zeta}_n)/2$. Then we have

$$\sigma_n := \sum_{\substack{0 \leqslant j \leqslant n/2 \\ j \overset{2}{\equiv} 0}} \frac{1}{(n-2j)!j!2^j} \alpha_2^{n-2j} \alpha_3^j, \qquad \overline{\sigma}_n = \sum_{\substack{0 \leqslant j \leqslant n/2 \\ j \overset{2}{\equiv} 1}} \frac{1}{(n-2j)!j!2^j} \alpha_2^{n-2j} \alpha_3^j.$$

First we claim that for any $1 \leqslant i \leqslant n/3$, there are constants $c_0, c_1, \ldots, c_{i-1} \in \mathbf{Q}$ such that

$$\mathrm{LT}(\sum_{j=0}^{i-1} c_j \alpha_2^j \overline{\sigma}_{n+i-j}) = \alpha_2^{n-3i+2} \alpha_3^{2i-1}. \tag{4.5}$$

Suppose $p_0(x) = 1$ and for $n \geqslant 1$, define $p_n(x)$ as the degree n polynomial $x(x+1) \cdot (x + n - 1)$. A straightforward computation shows that (4.5) is equivalent to finding c_j satisfying

$$\sum_{0 \leqslant j \leqslant i-1} c_j p_j(n+i-1-2k) = \begin{cases} 0 & 1 \leqslant k \leqslant 2i-1 \text{ and } k \overset{2}{\equiv} 1 \\ c & k = 2i-1 \end{cases} \tag{4.6}$$

for some non-zero constant c. Suppose M is the square matrix of size i such that for $0 \leqslant m, j \leqslant i-1$, the (m, j) entry of M is equal to $p_j(n+i-4m+1)$. The linear system in (4.6) has a solution if M is invertible. The determinant of M is equal to the determinant of the matrix M' whose (m, j) entry is $(n+i-4m+1)^j$. This can be seen by applying a sequence of column operations. Now the matrix M' is a Vandermonde matrix, and it is easily seen that it is invertible.

A similar argument shows that for $0 \leqslant i \leqslant n/3$, there are $d_0, d_1, \cdots, d_i \in \mathbf{Q}$ such that

$$\mathrm{LT}(\sum_{j=0}^{i} d_j \alpha_2^j \sigma_{n+i-j}) = \alpha_2^{n-3i} \alpha_3^{2i}. \tag{4.7}$$

We remark that the polynomial $\sum d_j \alpha_2^j \sigma_{n+i-j}$ in (4.7) is homogenous of degree $2n+2i$ with respect to the grading defined on $\mathbf{C}[\alpha_2, \alpha_3]$. Furthermore, all the monomials appearing in this polynomial have an even power of α_3. Similarly, the polynomial in (4.5) is homogenous of degree $2n+2i$ and it contains only monomials with odd powers of α_3.

Recursive formulas in (4.2) and the identity in (4.5) show that $\alpha_2^i \alpha_3^j$ with $2i + 3j \geqslant 2n$ and j being odd belongs to the leading term ideal of I_n^0. We cannot use (4.7) to treat the case that j is even because the polynomial in (4.7) contains the term σ_n which does not belong to I_n^0. However, if we replace σ_n in this sum with ζ_n, which is an element of I_n^0, we obtain:

$$\begin{aligned} d_i \alpha_2^i \zeta_n + \sum_{j=0}^{i-1} d_j \alpha_2^j \sigma_{n+i-j} =& b_1 \alpha_2^{n+i-2} \alpha_3 + b_3 \alpha_2^{n+i-6} \alpha_3^3 + \cdots + b_{2i-1} \alpha_2^{n-3i+2} \alpha_3^{2i-1} \\ &+ \alpha_2^{n-3i} \alpha_3^{2i} + \text{ lower order terms} \end{aligned} \tag{4.8}$$

All the monomials appearing in the first line of the right hand side are of the form $\alpha_2^i\alpha_3^j$ with $2i+3j \geqslant 2n$ and j being odd. In particular, using what we just proved for the monomials with odd powers of α_3, we can find constants $c'_0, \ldots, c'_{i-1}$ such that

$$\mathrm{LT}\left(\sum_{j=0}^{i-1} c'_j\alpha_1^j\overline{\sigma}_{n+i-j} - \sum_{k=1}^{i} b_{2k-1}\alpha_2^{n+i-4k+2}\alpha_3^{2k-1}\right) = k\alpha_2^{n-3i-2}\alpha_3^{2i+1} \tag{4.9}$$

for some constant k. Using (4.8) and (4.9), we have

$$\mathrm{LT}\left(d_i\alpha_2^i\zeta_n + \sum_{j=0}^{i-1} d_j\alpha_2^j\sigma_{n+i-j} - \sum_{j=0}^{i-1} c'_j\alpha_1^j\overline{\sigma}_{n+i-j}\right) = \alpha_2^{n-3i}\alpha_3^{2i}. \qquad \square$$

Define $\overline{I}_g$ as the image of the ideal $\widetilde{I}_g$ with respect to the homomorphism

$$\mathbf{A}_g^3 \to \mathbf{C}[\alpha_2, \alpha_3, \beta_2] \tag{4.10}$$

given by mapping β_3 and the curve classes ψ_r^i to 0. Since $\widetilde{I}_g$ includes β_3 and the curve classes, we have $\mathbf{A}_g^3/\widetilde{I}_g \cong \mathbf{C}[\alpha_2, \alpha_3, \beta_2]/\overline{I}_g$. Using (3.9)–(3.10), we have

$$\zeta_m^{g,k} \in \widetilde{I}_g \quad \text{if} \quad 0 \leqslant k \leqslant g, \quad m \geqslant 3g-k-1, \tag{4.11}$$

$$\overline{\zeta}_m^{g,k} \in \widetilde{I}_g \quad \text{if} \quad 0 \leqslant k \leqslant g, \quad m \geqslant 3g-k. \tag{4.12}$$

In the following proof, we slightly abuse notation and regard $\zeta_m^{g,k}$ and $\overline{\zeta}_m^{g,k}$ as elements of $\mathbf{C}[\alpha_2, \alpha_3, \beta_2]$ using the homomorphism (4.10).

Proposition 4.13. *The leading term ideal of $\overline{I}_g$ includes the monomials*

$$\{\alpha_2^i\alpha_3^j\beta_2^k \mid k \leqslant 2,\ 2i+3j+2k \geqslant 4g-2\} \cup \{\beta_2^3\}. \tag{4.14}$$

Proof. It follows from the definition of ζ_m and $\overline{\zeta}_m$ that

$$\zeta_m^{g,k}(-\alpha_2, -\alpha_3, 0) = \zeta_m \quad \text{and} \quad \overline{\zeta}_m^{g,k}(-\alpha_2, -\alpha_3, 0) = \overline{\zeta}_m. \tag{4.15}$$

Thus (4.11) and (4.12) with $k = g$ imply that ζ_{2g-1}, ζ_{2g}, $\overline{\zeta}_{2g}$ and $\overline{\zeta}_{2g+1}$ belong to $\overline{I}_g + (\beta_2)$. It follows from Proposition 4.3 that $\alpha_2^i\alpha_3^j$ with $2i+3j \geqslant 4g-2$ is in the leading term ideal of $\overline{I}_g$. From Lemma 3.33 and (4.15), we see that

$$\beta_2\zeta_{m-2} \in (\zeta_m^{g,g}, \zeta_m^{g,g-1}, \beta_2^2) \subset \overline{I}_g + (\beta_2^2), \quad \text{if} \quad m \geqslant 2g, \tag{4.16}$$

$$\beta_2\overline{\zeta}_{m-2} \in (\overline{\zeta}_m^{g,g}, \overline{\zeta}_m^{g,g-1}, \beta_2^2) \subset \overline{I}_g + (\beta_2^2), \quad \text{if} \quad m \geqslant 2g+1. \tag{4.17}$$

Therefore, another application of Proposition 4.3 implies that $\alpha_2^i\alpha_3^j\beta_2$ with $2i+3j \geqslant 4g-4$ is in the leading term ideal of $\overline{I}_g$. Finally, using Lemma 3.26 and (4.15), we see that

$$\beta_2^2\zeta_{m-1} \in (\zeta_{m+1}^{g,g}, \zeta_{m+2}^{g,g}, \zeta_{m+3}^{g,g}, \beta_2^3) \subset \overline{I}_g, \quad \text{if} \quad m \geqslant 2g-2, \tag{4.18}$$

$$\beta_2^2\overline{\zeta}_{m-1} \in (\overline{\zeta}_{m+1}^{g,g}, \overline{\zeta}_{m+2}^{g,g}, \overline{\zeta}_{m+3}^{g,g}, \beta_2^3) \subset \overline{I}_g, \quad \text{if} \quad m \geqslant 2g-1, \tag{4.19}$$

which shows that $\beta_2^2\zeta_{2g-3}$, $\beta_2^2\zeta_{2g-2}$, $\beta_2^2\overline{\zeta}_{2g-2}$ and $\beta_2^2\overline{\zeta}_{2g-1}$ belong to $\overline{I}_g$. Appealing to Proposition 4.3 again, we conclude that $\alpha_2^i\alpha_3^j\beta_2^2$ with $2i+3j \geqslant 4g-6$ is in the leading term ideal of $\overline{I}_g$. Finally β_2^3 is in the leading term ideal of $\overline{I}_g$ because it is an element of $\overline{I}_g$. □

Now we are almost ready to prove Theorem 2.22. We only need the following lemma, which can be proved in a straightforward way by induction.

Lemma 4.20. *For any non-negative integer n, let $f(n)$ denote the size of the following set:*

$$S_n := \{(i,j) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0} \mid 2i+3j < n\}.$$

If $n = 6k + r$ with $0 \leqslant r \leqslant 5$, then

$$f(n) = \frac{n^2 - r^2}{12} + 2\left\lfloor \frac{n}{6} \right\rfloor + f(r).$$

Moreover, $f(0) = 0$, $f(1) = 1$, and $f(r) = r - 1$ if we have $2 \leqslant r \leqslant 5$.

Proof of Theorem 2.22. By the isomorphism $\mathbf{A}_g^3/\widetilde{I}_g \cong \mathbf{C}[\alpha_2, \alpha_3, \beta_2]/\overline{I}_g$, it suffices to show that the cardinality of the set of monomials not in the leading term ideal of $\overline{I}_g$ is bounded above by $(2g-1)^2$. Lemma 4.13 implies this cardinality is bounded above by the size of:

$$\{(i,j,k) \in \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0} \times \mathbf{Z}_{\geqslant 0} \mid 2i+3j+2k < 4g-2,\ k \leqslant 2\}. \tag{4.21}$$

In the terminology of Lemma 4.20, the size of (4.21) is $f(4g-2) + f(4g-4) + f(4g-6)$. Since the set of mod 6 remainders of $4g-6, 4g-4, 4g-2$ is $\{0, 2, 4\}$, Lemma 4.20 implies

$$\begin{aligned} f(4g-2) + f(4g-4) + f(4g-6) &= \frac{(4g-2)^2}{12} + \frac{(4g-4)^2}{12} + \frac{(4g-6)^2}{12} \\ &\quad + 2\left(\left\lfloor \frac{2g-3}{3} \right\rfloor + \left\lfloor \frac{2g-2}{3} \right\rfloor + \left\lfloor \frac{2g-1}{3} \right\rfloor\right) + \frac{7}{3} \\ &= (2g-1)^2. \end{aligned}$$

□

5 Sutured instanton homology

In this section, we study sutured instanton homology for the gauge group $U(3)$. The construction is a slight variation of the one in [DX20]. Building on the work of that reference, we prove Theorem 1.5, which says that sutured instanton homology is well-defined, independent of the auxiliary choices in its construction. Following the strategy of [KM10b], we prove Theorem 1.6, a non-vanishing result for *taut* sutured manifolds. This is used to prove our topological applications, Theorems 1.2 and 1.3.

Suppose (Y, γ) is a 3-admissible pair and R is a connected surface of genus g such that $R \cdot \gamma \equiv 1 \pmod 3$. For a point y in Y, we generalize the notation from (2.4) and set

$$\beta_r = \mu_r(y), \qquad r \in \{2, 3\},$$

viewed as an endomorphism of $I_*^3(Y, \gamma)$. Analogous to (2.6), we can define an operator $\varepsilon(R)$ of degree -4 acting on $I_*^3(Y, \gamma)$ using the product cobordism $[-1, 1] \times Y$ with $U(3)$-bundle determined by the oriented 2-cycle $[-1, 1] \times \gamma \cup \{0\} \times R$. When the choice of R is clear from context, we write ε for $\varepsilon(R)$. We also have the operators $\mu_2(R)$ and $\mu_3(R)$ acting on $I_*^3(Y, \gamma)$, which can be defined without the assumption $R \cdot \gamma \equiv 1 \pmod 3$. All of these operators commute, and hence we can consider their simultaneous (generalized) eigenspaces.

Proposition 5.1. *Suppose* $(\lambda_2, \lambda_3, \eta_2, \eta_3)$ *is a simultaneous eigenvalue of the operators* $(\mu_2(R), \mu_3(R), \beta_2, \beta_3)$ *acting on* $I_*^3(Y, \gamma)$ *with* $\eta_2^3 = 27$ *and* $\eta_3 = 0$. *Then* $(\lambda_2, \lambda_3, \eta_2, \eta_3) \in \mathcal{E}_{g,1}^3$. *Moreover, the generalized eigenspace for any eigenvalue of the form* $(\pm\sqrt{3}\zeta^k(2g - 2), 0, 3\zeta^{2k}, 0)$ *agrees with the corresponding eigenspace.*

Proof. This proposition is the $U(3)$ analogue of [KM10b, Corollary 7.2] and can be verified in a similar way. We use functoriality to see that any relation among α_r and β_r in $V_{g,1}^3$ holds universally for any admissible pair (Y, γ) and an embedded surface R as above. To be more precise, let p be a polynomial with 4 variables such that $p(\alpha_2, \alpha_3, \beta_2, \beta_3)$ vanishes as an operator acting on $V_{g,1}^3$. Then we show that $p(\mu_2(R), \mu_3(R), \beta_2, \beta_3)$ vanishes as an operator acting on $I_*^3(Y, \gamma)$. This is sufficient to prove both claims in the statement of the proposition because they can be expressed in terms of polynomial relations among the operators $\mu_2(R)$, $\mu_3(R)$, β_2 and β_3, and then we can use the corresponding results in the special case of $V_{g,1}^3$ given in Theorem 2.14 and Proposition 2.23.

A regular neighborhood of $\{0\} \times R$ in the product cobordism $[-1, 1] \times Y$ can be used to decompose $[-1, 1] \times Y$ as the composition of cobordisms $D^2 \times R$ and W with three boundary components $-Y$, Y and $S^1 \times R$. This also induces a decomposition of γ where the intersection with $D^2 \times R$ can be assumed to be $D^2 \times \{x\}$ for $x \in R$. Suppose also that w is the induced 2-cycle on W. Then functoriality implies that for any polynomial p of 4 variables and any $v \in I_*^3(Y, \gamma)$ we have

$$p(\mu_2(R), \mu_3(R), \beta_2, \beta_3)(v) = I_*^3(W, w)(v \otimes p(\alpha_2, \alpha_3, \beta_2, \beta_3)(\mathbf{1})).$$

In particular, if $p(\alpha_2, \alpha_3, \beta_2, \beta_3)$ is a trivial operator acting on $V_{g,1}^3$, then the action of $p(\mu_2(R), \mu_3(R), \beta_2, \beta_3)$ on $I_*^3(Y, \gamma)$ is trivial. □

We define the instanton Floer homology group $I^3_*(Y,\gamma|R)$ as a simultaneous eigenspace for the point classes and the operators associated to the surface R in the following way:

$$I^3_*(Y,\gamma|R) = \ker(\mu_2(R) - \sqrt{3}(2g-2)) \cap \ker(\mu_3(R)) \cap \ker(\beta_2 - 3) \cap \ker(\beta_3). \quad (5.2)$$

In particular, equation (2.13) implies that

$$I^3_*(S^1 \times \Sigma_g, S^1 \times \{x\}|\Sigma_g) = \mathbf{C}. \quad (5.3)$$

If R' is disconnected, we modify (5.2) so that the intersection includes each of the operators $\mu_2(R') - \sqrt{3}(2g(R') - 2)$ and $\mu_3(R')$ for each connected component R' of R. In the case that (Y,γ) is the disjoint union of admissible pairs (Y_0,γ_1), (Y_1,γ_1) and $R \subset Y$ is given by $R_0 \sqcup R_1$ with $R_i \subset Y_i$, $R_i \cdot \gamma_i \equiv 1$ mod 3, then we define

$$I^3_*(Y,\gamma|R) = I^3_*(Y_0,\gamma_1|R_0) \otimes I^3_*(Y_1,\gamma_1|R_1).$$

This can be extended to more than two connected components in the same way.

Remark 5.4. In [DX20], the instanton homology group $I^3_*(Y,\gamma|R)$ is defined by taking the simultaneous generalized kernel of the operators in (5.2) and the operator $\varepsilon - 1$. Proposition 5.1 shows that the generalized kernel for the operators in (5.2) agrees with the ordinary kernel. Furthermore, we show in the proof of Proposition 2.23 that in the case of $(S^1 \times \Sigma_g, S^1 \times \{x\})$ any element in (5.2) already belongs to the kernel of $\varepsilon - 1$. Therefore, the proof of Proposition 5.1 shows that the same claim holds for an arbitrary pair (Y,γ). As a consequence of these observations, our definition in (5.2) agrees with that of [DX20].

Proposition 5.5. *Suppose S is an embedded surface in Y. Then the operators $\mu_2(S)$ and $\mu_3(S)$ preserve the subspace $I^3_*(Y,\gamma|R)$ of $I^3_*(Y,\gamma)$. Furthermore, if (λ_2,λ_3) is a simultaneous eigenvalue of $(\mu_2(S),\mu_3(S))$, then there are $a,b \in \mathbf{Z}$ with $a \equiv b \pmod 2$ such that $(\lambda_2,\lambda_3) = (\sqrt{3}a, \sqrt{3}ib)$ and*

$$|a| + |b| \leqslant 2g(S) - 2. \quad (5.6)$$

This proposition is the counterpart of [KM10b, Proposition 7.5]. However, the proof there seems to require some modifications, even in the case $N = 2$. The modification used in the following proof was communicated to us by Peter Kronheimer.

Proof. In the case that $S \cdot \gamma \equiv 1 \pmod 3$, the claim follows from Proposition 5.1 and the case $S \cdot \gamma \equiv -1 \pmod 3$ can be verified in a similar way. Using a topological trick, the case $S \cdot \gamma \equiv 0 \pmod 3$ can be also reduced to the previous cases. Suppose $v \in I^3_*(Y,\gamma|R)$ is a simultaneous eigenvector of $(\mu_2(S),\mu_3(S))$ with eigenvalues (λ_2,λ_3). Suppose σ_n is the homology class $n[S]+[R]$, which is represented by a connected surface S_n in Y. Since the μ operators depend only on the homology classes of the involved surfaces, v is a simultaneous eigenvector of $(\mu_2(S_n),\mu_3(S_n))$ with eigenvalues $(n\lambda_2 + \sqrt{3}(2g-2), n\lambda_3)$. We have $S_n \cdot \gamma \equiv 1 \pmod 3$, which in the case that $n = 1$ implies that $(\lambda_2,\lambda_3) = (\sqrt{3}a, \sqrt{3}ib)$ for some integers a and b with the same parity.

Next, to show that (a,b) satisfies (5.6), we need some control on the genus of the connected surface S_n. In fact, it suffices to find an embedded surface S_n in $[-1,1]\times Y$ with the same homology class. Take a cyclic n-sheeted covering $\widetilde{S}$ of S. It is straightforward to see that $\widetilde{S}$ can be embedded in $D^2\times S$ in such a way that the composition of this embedding with the projection map $D^2\times S\to S$ is the covering projection $\widetilde{S}\to S$. In particular, the genus of $\widetilde{S}$ is equal to $n(g(S)-1)+1$. The embedding of $\widetilde{S}$ in $D^2\times S$ induces an embedding of this surface in a neighborhood of $\{0\}\times S\subset[-1,1]\times Y$ realizing the homology class $n[S]$. By tubing this surface and a disjoint copy of R, we obtain a connected surface S_n of genus $n(g(S)-1)+g+1$ with the homology class $n[S]+[R]$. Since $S_n\cdot\gamma\equiv 1 \pmod 3$ (and the self-intersection of S_n is trivial), we have

$$|na+2g-2|+|nb|\leqslant 2n(g(S)-1)+2g.$$

Diving by n and taking $n\to\infty$ gives (5.6). □

Remark 5.7. For a genus one surface T, the group $I^3_*(S^1\times T, S^1\times\{x\})$ is 3-dimensional and hence it splits as the sum of 1-dimensional eigenspaces for the three simultaneous eigenvalues in $\mathcal{E}^3_{1,1}$. In particular, the actions of $\mu_2(T)$ and $\mu_3(T)$ are trivial on $I^3_*(S^1\times T, S^1\times\{x\})$. Using a similar argument as in the proof of Proposition 5.1, we can see more generally that if (Y,γ) is an admissible pair and T is an embedded surface of genus 1 in Y with $\gamma\cdot T\equiv 1$ mod 3, then the actions of $\mu_2(T)$ and $\mu_3(T)$ are trivial. In particular, we have

$$I^3_*(Y,\gamma|T)=\ker(\beta_2-3)\cap\ker(\beta_3).$$

Similar to [KM10b, Corollary 7.6], we consider the action of $(\mu_2(\sigma),\mu_3(\sigma))$ on $I^3_*(Y,\gamma|R)$ for all homology classes $\sigma\in H_2(Y;\mathbf{Z})$ to obtain a splitting of $I^3_*(Y,\gamma|R)$ as

$$I^3_*(Y,\gamma|R)=\bigoplus_s I^3_*(Y,\gamma|R;s) \tag{5.8}$$

where the direct sum is over all homomorphisms

$$s: H_2(Y;\mathbf{Z})\to\Gamma\subset\mathbf{Z}\oplus\mathbf{Z}$$

with Γ being the sublattice of $\mathbf{Z}\oplus\mathbf{Z}$ given by pairs (a,b) with $a\equiv b \pmod 2$. For $s=(s_2,s_3)$ as above, the summand $I^3_*(Y,\gamma|R;s)$ is given as

$$\bigcap_{\sigma\in H_2(Y;\mathbf{Z})}\bigcup_{N\geqslant 0}\Big(\ker\Big(\mu_2(\sigma)-\sqrt{3}s_2(\sigma)\Big)^N\cap\ker\Big(\mu_3(\sigma)-\sqrt{3}is_3(\sigma)\Big)^N\Big).$$

As a corollary of Proposition 5.5, for any $\sigma\in H_2(Y;\mathbf{Z})$ with a surface representative S of genus g, the summand $I^3_*(Y,\gamma|R;s)$ can be non-trivial only if

$$\|s(\sigma)\|_1\leqslant 2g(S)-2.$$

Here $\|\cdot\|_1$ denotes the L^1 norm of vectors in $\mathbf{R}^2$.

The sutured instanton Floer homology group $SHI^3_*(M, \alpha)$ is defined with the aid of the instanton Floer homology groups in (5.2) for any *balanced* sutured manifold (M, α). Following [Gab83, Juh06], a balanced sutured manifold (M, α) consists of an oriented 3-manifold M without any closed component and a collection of oriented simple closed curves α in the boundary of M. The boundary of M is decomposed into three parts

$$\partial M = A(\alpha) \cup R_+(\alpha) \cup R_-(\alpha),$$

where $A(\alpha)$ is the closure of a tubular neighborhood of α. The connected components of $\partial M \backslash A(\alpha)$ are oriented, and $R_+(\alpha)$ (resp. $R_-(\alpha)$) is the union of such connected components whose orientation is given by the outward-normal-first convention (resp. inward-normal-first convention). The 2-dimensional manifolds $R_\pm(\alpha)$ do not have any closed connected component and the induced orientation on any of their boundary components (using outward-normal-first convention) agrees with the orientation of the corresponding suture. (Note that this condition fixes the orientation of the connected components of $\partial M \backslash A(\alpha)$.) Finally we require that $\chi(R_+(\alpha)) = \chi(R_-(\alpha))$.

Example 5.9. (Product sutured manifolds) Let $F_{g,k}$ denote the oriented surface of genus g with $k \geqslant 1$ boundary components. Then $M = [-1, 1] \times F_{g,k}$ and $\alpha = \{0\} \times \partial F_{g,k}$ give a balanced sutured manifold with $R_\pm(\alpha) = \{\pm 1\} \times F_{g,k}$ and $A(\alpha) = [-1, 1] \times \partial F_{g,k}$.

Example 5.10. Any closed oriented 3-manifold Y with a basepoint can be used to produce a sutured manifold $(Y(1), \alpha(Y))$, where $Y(1)$ is the complement of a ball neighborhood of the basepoint in Y and $\alpha(Y)$ is a simple closed curve in the boundary of $Y(1)$. Any knot K in a 3-manifold Y can be used to produce a sutured manifold $(Y(K), \alpha(K))$ where $Y(K)$ is the exterior of K and $\alpha(K)$ consists of two meridional simple closed curves.

The closure of a balanced sutured manifold (M, α) is a closed 3-manifold Z_α, defined in the following way. Suppose the number of sutures is equal to k, and consider the product sutured manifold $[-1, 1] \times F_{g,k}$ for an arbitrary g. Gluing the neighborhood $A(\alpha)$ of the sutures in ∂M to $[-1, 1] \times \partial F_{g,k}$ determines a 3-manifold Z^0_α with two boundary components $\overline{R}_+$ and $\overline{R}_-$. The surface $\overline{R}_\pm$ is the union of $R_\pm$ and $\{\pm 1\} \times F_{g,k}$. Since (M, α) is balanced, $\overline{R}_+$ and $\overline{R}_-$ are connected oriented surfaces of the same genus. We pick an orientation-preserving diffeomorphism $\varphi : \overline{R}_+ \to \overline{R}_-$ to identify these two boundary components, obtaining the closure Z_α.

The surfaces $\overline{R}_\pm$ determine a closed surface $\overline{R} \subset Z_\alpha$. We require that there is a simple closed curve c in $F_{g,k}$, that gives rise to non-separating curves in $\overline{R}_\pm$ and the gluing map φ maps these curves to each other. (This can always be arranged, for example, by taking $g \geqslant 1$ and setting c to be a non-separating oriented simple close curve in $F_{g,k}$.) The curve c determines a non-separating closed curve in $\overline{R}$, which is still denoted by c. In particular, we may fix another oriented simple closed curve c' in $\overline{R}$ intersecting c transversely at one point. By fixing a basepoint $x \in F_{g,k}$ and demanding that $\varphi(x) = x$, we obtain a curve $\gamma \subset Z_\alpha$ from $[-1, 1] \times \{x\} \subset Z^0_\alpha$. The $U(3)$ sutured instanton homology of (M, α) is defined as

$$SHI^3_*(M, \alpha) := I^3_*(Z_\alpha, \gamma|\overline{R}).$$

We now prove Theorem 1.5, which says that this sutured homology group is an invariant of (M, α), i.e. it does not depend on the choice of g nor the gluing map φ.

Proof of Theorem 1.5. A version of excision for instanton Floer homology groups $I_*^3(Y, \gamma|R)$ is proved in [DX20, Theorem 5.16], and is used to show that $SHI_*^3(M, \alpha)$ is independent of the gluing map φ. Using the excision theorem in [DX20] and Theorem 2.14, we show independence from g following the argument in [KM10b]. This requires a further understanding of the instanton homology of $S^1 \times \Sigma_g$ for different $U(3)$ bundles over this manifold. In the following, let c_0 and c_0' be non-separating oriented simple closed curves in Σ_g that have exactly one transversal intersection point. The curve c_0 determines the 2-dimensional torus $T = S^1 \times c_0$ in $S^1 \times \Sigma_g$. By fixing a basepoint in S^1, we may regard c_0' as a 1-cycle in $S^1 \times \Sigma_g$. We also write γ_1 for the 1-cycle $S^1 \times \{x\}$ in Σ_g.

First consider the instanton Floer homology group $B := I_*^3(S^1 \times \Sigma_g, \gamma_1 + c_0'|\Sigma_g)$. Applying the excision result of [DX20, Theorem 5.16] twice in the same way as in the proof of [KM10b, Proposition 7.8], we obtain an isomorphism

$$B \otimes B \otimes B \cong I_*^3(S^1 \times \Sigma_g, \gamma_1 + 3c_0'|\Sigma_g). \tag{5.11}$$

As the Floer groups $I_*^3(Y, \gamma)$ depend only on the element of $H^2(Y; \mathbf{Z}/3)$ induced by γ, the right side of (5.11) is isomorphic to $\mathbf{C}$ by (5.3). Therefore, B is also 1-dimensional.

Next, we consider the instanton Floer homology group $I_*^3(S^1 \times \Sigma_g, c_0'|T)$. The genus one version of the excision theorem of [DX20, Theorem 5.16] implies that

$$I_*^3(S^1 \times \Sigma_g, c_0'|T) \otimes I_*^3(S^1 \times \Sigma_1, \gamma_1 + c_0'|T) \cong I_*^3(S^1 \times \Sigma_g, \gamma_1 + c_0'|T). \tag{5.12}$$

The excision isomorphism intertwines the action of $\mu_i(\Sigma_g) + \mu_i(\Sigma_1)$ on the left hand side of (5.12) and the action of $\mu_i(\Sigma_g)$ on the right hand side. This follows from the fact that the excision isomorphism is given by a homomorphism associated to a cobordism

$$W : S^1 \times \Sigma_g \sqcup S^1 \times \Sigma_1 \to S^1 \times \Sigma_g$$

and the homology class $[\Sigma_g] + [\Sigma_1]$ induced from the incoming end and $[\Sigma_g]$ from the outgoing end are homologous on W. According to Remark 5.7, the action of $\mu_i(\Sigma_1)$ is trivial and hence $I_*^3(S^1 \times \Sigma_g, c_0'|T)$ and $I_*^3(S^1 \times \Sigma_g, \gamma_1 + c_0'|T)$ are isomorphic as modules over $\mathbf{Q}[\mu_2(\Sigma_g), \mu_3(\Sigma_g)]$. In particular, this shows that the simultaneous eigenvalues of the operators $(\mu_2(\Sigma_g), \mu_3(\Sigma_g))$ acting on $I_*^3(S^1 \times \Sigma_g, c_0'|T)$ are of the form $(\sqrt{3}a, \sqrt{3}ib)$ with $|a| + |b| \leqslant 2g - 2$ and the $(\sqrt{3}(2g-2), 0)$-eigenspace is 1-dimensional.

Now, let Z_α be a closure of (M, α) given by the surface $F_{g,k}$ and a gluing map φ. Replacing $F_{g,k}$ with $F_{g+1,k}$ and stabilizing φ in the obvious way determines a different closure Z_α'. We also write $\overline{R}$ and $\overline{R}'$ for the distinguished surfaces in Z_α and Z_α' whose genera are related by $g(\overline{R}') = g(\overline{R}) + 1$. These closed curves c and c' in $\overline{R}$ determine two oriented simple closed curves in $\overline{R}'$ which we still denote by c and c'.

To prove our claim, we need to show that

$$I_*^3(Z_\alpha, \gamma|\overline{R}) \cong I_*^3(Z_\alpha', \gamma|\overline{R}'). \tag{5.13}$$

By applying the excision result in [DX20, Theorem 5.16] for the copies of the surface $\overline{R}$ in the two admissible pairs (Z_α, γ), $(S^1 \times \overline{R}, \gamma_1 + c')$ and using the 1-dimensionality of the latter vector space, we conclude that

$$I^3_*(Z_\alpha, \gamma|\overline{R}) \cong I^3_*(Z_\alpha, \gamma + c'|\overline{R}).$$

Thus, to show (5.13), it suffices to verify that

$$I^3_*(Z_\alpha, \gamma + c'|\overline{R}) \cong I^3_*(Z'_\alpha, \gamma + c'|\overline{R}'). \tag{5.14}$$

By our assumption on c and the gluing map φ, there are copies of $T = S^1 \times c$ in Z_α and Z'_α. Another application of [DX20, Theorem 5.16] similar to (5.12) implies that

$$I^3_*(S^1 \times \Sigma_2, c'_0|T) \otimes I^3_*(Z_\alpha, \gamma + c'|T) \cong I^3_*(Z'_\alpha, \gamma + c'|T),$$

and this isomorphism intertwines the action of $\mu_i(\Sigma_2) + \mu_i(\overline{R})$ on the left hand side and the action of $\mu_i(\overline{R}')$ on the right hand side. Combining this fact, Remark 5.7, Proposition 5.5 and our analysis of the instanton Floer homology group $I^3_*(S^1 \times \Sigma_2, c'_0|T)$ verifies the claimed isomorphism in (5.14). □

From the proof of Theorem 1.5 one can see that the above construction can be generalized in various directions. First, one can consider non-trivial $U(3)$ bundles on sutured manifolds. More precisely, let (M, α) be a sutured manifold and w be a properly embedded oriented curve in M such that w is disjoint from $A(\alpha)$ and the intersection of w with $R_\pm(\alpha)$ is a collection of points $\pi_\pm = \{p_1^\pm, \dots, p_k^\pm\}$ such that the intersection of w with $R_\pm(\alpha)$ at the points $p_i^\pm$ have the same sign. In forming the closure Z_α of (M, α), we require that the gluing map sends the point p_i^+ to p_i^-. Thus we obtain a closed oriented curve $\overline{w}$. We define the sutured instanton homology $SHI^3_*(M, \alpha)_w$ of (M, α, w) as the Floer homology group $I^3_*(Z_\alpha, c' + \overline{w}|\overline{R})$. In particular, the instanton Floer homology of a product sutured manifold for any choice of w is still 1-dimensional.

Following the same proof as that of Theorem 1.5, we see that this Floer homology group is independent of the specific choice of the closure and is also isomorphic to the instanton homology groups $I^3_*(Z_\alpha, d \cdot \gamma + \overline{w}|\overline{R})$ and $I^3_*(Z_\alpha, d \cdot \gamma + c' + \overline{w}|\overline{R})$ where for the former instanton homology group we need that $d + \overline{w} \cdot \overline{R} \not\equiv 0 \bmod 3$. It is straightforward to see that the isomorphism class of $SHI^3_*(M, \alpha)_w$ depends only on the homeomorphism type of (M, α) and the isomorphism type of the $U(3)$-bundle on M determined by w. Furthermore, for any $U(3)$-bundle on M one can arrange w satisfying the above requirements.

We can also see from the proof of Theorem 1.5 that to form the closure of a sutured manifold (M, α) we do not necessarily need to use a connected sutured manifold $[-1, 1] \times F_{g,k}$. We can use a product sutured manifold $[-1, 1] \times F$ as long as $\overline{R}$ is connected and each connected component of F has a simple closed curve that becomes a non-separating curve in $\overline{R}$. This flexibility in forming the closure will be useful below. Finally, as another consequence of Theorem 1.5, we make the following observation.

Lemma 5.15. *Gluing a product 1-handle to a sutured manifold (M, α, w) along its sutures does not change the isomorphism type of $SHI^3_*(M, \alpha)_w$.*

A product 1-handle is the product $[-1,1]\times H$ where H is the 2-dimensional 1-handle given as $I\times I$ for an interval I. Fixing an embedding of $I\times\partial I$ into α determines an embedding of $[-1,1]\times I\times\partial I$ into $A(\alpha)$. Now to glue the product 1-handle $[-1,1]\times H$ to (M,α,w), we identify part of the boundary of the product 1-handle given by $[-1,1]\times I\times\partial I$ with its image in $A(\alpha)$ via the embedding. We assume that this gluing is done in a way that the resulting 3-manifold is orientable. With this assumption, the resulting 3-manifold admits the structure of a sutured manifold in an obvious way.

Proof. Suppose (M',α',w) is obtained by gluing a product 1-handle to (M,α,w). A closure of (M',α',w), obtained by gluing the product sutured manifold $[-1,1]\times F_{g,d}$ to M', can be regarded as a closure of (M,α), where we use the product sutured manifold $[-1,1]\times(F_{g,d}\cup H)$ in forming the closure. From this one can easily see that the sutured instanton homologies of (M,α,w) and (M',α',w') are isomorphic to each other. $\square$

The operation of *surface decomposition* can be used to simplify sutured manifolds [Gab83]. A *decomposing surface* S in a balanced sutured manifold (M,α) is a properly oriented surface S in M such that any connected component of $\partial S\cap A(\gamma)$ is either a properly embedded non-separating arc in $A(\gamma)$ or a simple closed curve oriented in the same sense as the suture in the corresponding connected component of $A(\gamma)$. Removing a small tubular neighborhood $N(S)$ of S from M produces a new sutured manifold (M',α') with

$$A(\alpha')=\big(A(\alpha)\cap\partial M'\big)\cup N_{\partial M'}(S_+\cap R_-(\alpha))\cup N_{\partial M'}(S_-\cap R_+(\alpha)),$$
$$R_\pm(\alpha')=\big(R_\pm(\alpha)\cap M'\big)\cup S_\pm\backslash\mathrm{int}(A(\alpha'))$$

where, after identifying $N(S)$ with $[-1,1]\times S$ as an oriented 3-manifold, $S_\pm$ is given by $\{\mp1\}\times S\subset\partial N(S)\cap M'$. This operation of surface decomposition is usually denoted

$$(M,\alpha)\overset{S}{\rightsquigarrow}(M',\alpha').$$

We may extend this definition in an obvious way in the presence of non-trivial bundle data w. If w is a properly oriented simple closed curve intersecting S and its boundary transversely, then the intersection w' of w with M' determines a properly embedded oriented curve in M' with the required properties. In this case, we write

$$(M,\alpha,w)\overset{S}{\rightsquigarrow}(M',\alpha',w').$$

Theorem 2.14 allows us to prove an analogue of surface decomposition theorems in [Juh08, KM10b] for our version of instanton Floer homology.

Proposition 5.16. *Suppose S is a decomposing surface for a sutured manifold (M,α,w). Assume that S does not have any closed components, and for every connected component V of $R_\pm(\alpha)$, the set of closed components of $\partial S\cap V$ consist of parallel oriented boundary-coherent simple closed curves. Suppose (M',α',w') is the sutured manifold obtained from decomposing along S. Then $SHI^3_*(M',\alpha')_{w'}$ is a summand of $SHI^3_*(M,\alpha)_w$.*

An oriented simple closed curve c in an oriented surface V is boundary coherent if either c is non-separating or removing c from V gives a disconnected surface with a connected component V_0 whose only boundary component is c. In the latter case, we require that the orientation of c is given by the outward-normal-first convention applied to V_0.

Proof. We follow a similar argument as in the proof of Theorem [KM10b, Proposition 6.9 and Proposition 7.11]. Without loss of generality, we can assume S is connected. We can also assume that all connected components of ∂S have non-empty intersection with $R_{\pm}(\alpha)$ using [Juh08, Lemma 4.5]. Next, we glue product 1-handles to (M, α) and S as in the proof of [KM10b, Proposition 6.9] to obtain a decomposing surface in a sutured manifold where ∂S consists of simple closed curves $C_1^{\pm}, \dots\ C_{n_{\pm}}^{\pm}$ in $R_{\pm}(\alpha)$. Lemma 5.15 implies that proving the claim for this new sutured manifold and the decomposing surface implies the claim for the original surface decomposition.

The closed curves $C_i^{\pm}$ determine linearly independent homology classes in $H_1(R_{\pm}(\alpha))$. If $n_+ \neq n_-$, we may apply further finger moves as in [Juh08, Lemma 4.5] and then glue product 1-handles as in [KM10b, Proposition 6.9] to increase the number of the boundary components of ∂S in one of $R_{\pm}(\alpha)$ while preserving the number of such components in the other one. Thus we may assume $n_+ = n_-$. In summary, the boundary of our decomposing surface satisfies similar assumptions as in [KM10b, Lemma 6.10].

In order to form a closure of (M, α, w), first we glue a product sutured manifold $[-1, 1] \times F_{g,d}$ to M along $A(\alpha)$. The two boundary components $\overline{R}_{\pm}(\alpha)$ of the resulting 3-manifold contains the curves $C_i^{\pm}$ which are still linearly independent in $H_1(\overline{R}_{\pm}(\alpha))$. In particular, we can pick a diffeomorphism $\varphi : \overline{R}_+(\alpha) \to \overline{R}_-(\alpha)$, which maps C_i^+ to C_i^- in an orientation-reversing way. By forming the closure Z_α of (M, α, w) via φ we obtain closed oriented connected surfaces $\overline{R}$ and $\overline{S}$ induced by $\overline{R}_{\pm}(\alpha)$ and S. Moreover, these two surfaces intersect in a collection of simple closed curves C_i that are induced by $C_i^{\pm}$. By smoothing out these intersection curves we obtain another closed oriented connected surface F in the same homology class as $[\overline{R}] + [\overline{S}]$. We assume that $\overline{w} \cdot \overline{R} \equiv 0$ mod 3. Then

$$SHI_*^3(M, \alpha)_w = I_*^3(Z_\alpha, \gamma + \overline{w}|\overline{R}),$$

where γ is induced by a point in $F_{g,d}$ in the same way as before and $\overline{w}$ is the closure of w. The proof in the case $\overline{w} \cdot \overline{R} \not\equiv 0 \pmod 3$ is similar, as we can replace the 1-cycle γ with some other multiple of it to define the instanton Floer homology of (M, α, w).

The key observation of [KM10b, Lemma 6.10] is that Z_α can be also regarded as a closure for (M', α', w') where the counterpart of the surface $\overline{R}$ is F. It can be easily seen that the closure of w' is still $\overline{w}$. To be more precise, there is a disconnected surface T without any closed component such that after gluing the product sutured manifold $[-1, 1] \times T$ to (M', α', w') and picking an appropriate gluing map φ' we obtain a 3-manifold diffeomorphic to Z_α together with the surface F and the 1-cycle $\overline{w}$. The disconnected surface T satisfies the required property mentioned above such that it can be used to define $SHI_*^3(M', \alpha')_{w'}$. In particular, this sutured instanton Floer homology group is isomorphic to $I_*^3(Z_\alpha, \gamma + \overline{w}|F)$.

Let $v \in I_*^3(Z_\alpha, \gamma|F)$ be a simultaneous eigenvector for the action of the operators $(\mu_2(\overline{R}), \mu_3(\overline{R}))$ with eigenvalues (λ_2, λ_3). Since $\overline{R} \cdot \gamma = 1$, Proposition 5.5 implies that

$(\lambda_2, \lambda_3) = (\sqrt{3}a, \sqrt{3}ib)$ with a and b of the same parity and

$$|a| + |b| \leqslant 2g(\overline{R}) - 2. \tag{5.17}$$

We also have $[F] = [\overline{R}] + [\overline{S}]$, $\chi(F) = \chi(\overline{R}) + \chi(\overline{S})$, which implies that v is a simultaneous eigenvector for the action of $(\mu_2(\overline{S}), \mu_3(\overline{S}))$ with eigenvalues $(\sqrt{3}(2g(F) - 2) - \lambda_2, -\lambda_3)$. We apply Proposition 5.5 again to get a bound on the norm of these eigenvalues:

$$|2g(F) - 2 - a| + |b| \leqslant 2g(\overline{S}) - 2, \tag{5.18}$$

The inequalities in (5.17) and (5.18) imply that $(a, b) = (2g(\overline{R}) - 2, 0)$. As a result, the only simultaneous eigenvalue of $(\mu_2(\overline{R}), \mu_3(\overline{R}))$ acting on $v \in I^3_*(Z_\alpha, \gamma|F)$ is $(\sqrt{3}(2g(\overline{R}) - 2), 0)$. This in turn implies that $I^3_*(Z_\alpha, \gamma|F)$ is the summand of $I^3_*(Z_\alpha, \gamma|\overline{R})$ given by the simultaneous eigenspace of the operators $(\mu_2(\overline{S}), \mu_3(\overline{S}))$ corresponding to the eigenvalue $(\sqrt{3}(2g(\overline{S}) - 2), 0)$. In particular, $SHI^3_*(M', \alpha')$ is a summand of $SHI^3_*(M, \alpha)$. □

Recall that a sutured manifold (M, α) is *taut* if M is irreducible and $R_+(\alpha)$, $R_-(\alpha)$ are *norm minimizing* in their homology classes in $H_2(M, A(\gamma))$ [Gab83, Definition 2.4]. (In general, if Y is a 3-manifold and Z is a codimension 0 submanifold of ∂Y, then an embedding $(S, \partial S)$ into (Y, Z) for a surface S is norm minimizing if S is incompressible and S realizes the Thurston norm of the homology class $[S] \in H_2(Y, Z)$.) If (M, α) is taut, then we say (M, α, w) is taut for any choice of a 1-cycle w.

Corollary 5.19. *For any balanced taut sutured manifold (M, α, w), the sutured instanton homology group $SHI^3_*(M, \alpha)_w$ is non-trivial.*

Proof. Following the proof of [Juh08, Theorem 1.4], there is a sequence of decompositions

$$(M, \alpha) \overset{S_1}{\rightsquigarrow} (M_1, \alpha_1) \overset{S_2}{\rightsquigarrow} \cdots \overset{S_n}{\rightsquigarrow} (M_n, \alpha_n) \tag{5.20}$$

such that each S_i satisfies the assumptions in Proposition 5.16 and (M_n, α_n) is a product sutured manifold. Now the claim follows from Proposition 5.16 and the fact that sutured instanton homology of a product sutured manifold for any $U(3)$-bundle is 1-dimensional. □

Corollary 5.21. *Suppose Y is an irreducible 3-manifold, γ is a 1-cycle in Y and R is a norm minimizing embedded surface in Y. Then $I^3_*(Y\#T^3, \gamma + \gamma_1|R\#T^2)$ is non-trivial, where γ_1 is the 1-cycle in T^3 given by $S^1 \times \{x\}$ with $x \in T^2$.*

Proof. Cutting (Y, γ) along R produces a 3-manifold with two boundary components R_+ and R_-, which are copies of R. Glue a 1-handle to this 3-manifold along the discs $D_\pm \subset R_\pm$ which correspond to a fixed disc $D \subset R$. The resulting 3-manifold M is a balanced sutured manifold with one suture α and the complement of an annular neighborhood of the suture in the boundary is given by the surfaces $R_\pm \backslash D_\pm$. The 1-cycle γ induces a 1-cycle w in the sutured manifold (M, α). We may also regard M is a submanifold of Y. In particular, the properly embedded surfaces $R_\pm \backslash D_\pm$ in $(M, A(\alpha))$ are norm minimizing because R is norm minimizing in Y. Furthermore, if M is reducible, then the irreducibility of Y implies that R

can be embedded in a ball in Y which contradicts the assumption that R is norm minimizing. Thus (M, α, w) is taut, and hence $SHI^3_*(M, \alpha, w)$ is non-trivial.

We take a closure of (M, α, w) by gluing $[-1, 1] \times F_{1,1}$ and then gluing the two boundary components of the resulting 3-manifold in the obvious way. The resulting closure can be identified with $Y \# T^3$ with the distinguished embedded surface $R \# T^2$ and the 1-cycle $\overline{w} = \gamma$. In particular, $SHI^3_*(M, \alpha)_w$ is equal to $I^3_*(Y \# T^3, \gamma + \gamma_1 | R \# T^2)$. □

Remark 5.22. A similar proof can be used to show that $I^2_*(Y \# T^3, \gamma + \gamma_1 | R \# T^2) \neq 0$. In the case that (Y, γ) is 2-admissible, combining this with the connected sum theorems of instanton Floer homology in the admissible case [Sca15], one can see that $I^2_*(Y, \gamma | R)$ is also non-trivial. This is essentially the same non-vanishing result as in [KM10b, Theorem 7.21]. However, it seems that one needs to modify the statement and the proof of [KM10b, Theorem 7.21]. It is reasonable to expect that there is a connected sum theorem for $U(3)$ instanton Floer homology which implies that $I^3_*(Y \# T^3, \gamma + \gamma_1 | R \# T^2)$ is non-trivial only if $I^3_*(Y, \gamma | R) \neq 0$ whenever (Y, γ) is 3-admissible. In Section 7, we show that $I^3_*(Y, \gamma)$ is non-trivial using a non-vanishing result for symplectic 4-manifolds.

Proof of Theorem 1.3. Suppose γ is a 1-cycle representing the Poincaré dual of ω. To give a representation $\rho : \pi_1(Y) \to PU(3)$ satisfying the required property, it suffices to find a projectively flat connection on a $U(3)$-bundle over Y with $c_1 = \mathrm{PD}(\gamma)$. Furthermore, we may assume that Y is prime. If Y is a rational homology sphere, then there is a flat $U(1)$-connection on Y whose first Chern class is $\mathrm{PD}(\gamma)$. By taking the sum of this connection and the trivial $SU(2)$ connection, we obtain a $U(3)$ flat connection with the required property. If $Y = S^1 \times S^2$, then the assumption implies that ω is trivial and we may take the trivial flat connection. Otherwise Y is irreducible with positive b_1 and Corollary 5.21 implies that $I_*(Y \# T^3, \gamma + \gamma_1 | R)$ is not zero, where R is a norm minimizing embedded surface in Y (representing a non-trivial homology class). In particular, there exists a projectively flat connection on the $U(3)$-bundle over Y with $c_1 = \mathrm{PD}(\gamma)$. This gives a representation $\rho : \pi_1(Y) \to PU(3)$ satisfying the claim. □

For a knot K in a 3-manifold Y, the $U(3)$ instanton knot homology of (Y, K), denoted by $KHI^3_*(Y, K)$, is defined to be $SHI^3_*(Y(K), \alpha(K))$, where $(Y(K), \alpha(K))$ is the sutured manifold of Example 5.10. As explained in [KM10b], a closure of $(Y(K), \alpha(K))$ is given by $Z(K)$, the 3-manifold obtained by gluing $S^1 \times F_{1,1}$ to the exterior of K such that $S^1 \times \{x\}$ is mapped to a meridian of K for any $x \in \partial F_{1,1}$. Let c and c' be two simple closed curves in $F_{1,1}$ intersecting transversely in exactly one point and $T = S^1 \times c$. Then

$$KHI^3_*(Y, K) = I^3_*(Z(K), c' | T). \tag{5.23}$$

This instanton knot homology group is isomorphic to $I^3_*(Z(K), \gamma + c' | T)$ where γ is the 1-cycle $S^1 \times \{x\}$ for some $x \in F_{1,1}$. Now if K is null-homologous, then we can pick the gluing map in the definition of $Z(K)$ so that $\{pt\} \times \partial F_{1,1}$ is glued to a longitude of K. In this case, we can glue a Seifert surface S of genus g to $F_{1,1}$ and obtain an embedded surface $\overline{S}$ in $Z(K)$ of genus $g + 1$. In particular, $(\mu_2(S), \mu_3(S))$ gives a pair of operators acting on $KHI^3_*(Y,K)$. The simultaneous generalized eigenspace decomposition with respect to the action of these operators determines a splitting of $KHI^3_*(Y,K)$ given as follows, that depends only on the homology class of $\overline{S}$:

$$KHI^3_*(Y,K) = \bigoplus_{(a,b)\in\mathcal{C}_{g+1}} KHI^3_*(Y,K;a,b), \tag{5.24}$$

where $KHI^3_*(Y,K;a,b)$ is the generalized eigenspace of $(\mu_2(S),\mu_3(S))$ for the eigenvalues $(\sqrt{3}a,\sqrt{3}ib)$. To limit the possible eigenvalues appearing in this decomposition, we have used Proposition 5.5. The decomposition (5.24) will be discussed further in Section 9.

Proof of Theorem 1.2. Suppose S is a Seifert surface of minimal genus for the knot K. Then the decomposition of $(Y(K),\alpha(K))$ along S determines a sutured manifold $(Y(S),\alpha(S))$. It is shown in the proof of [DX20, Proposition 5.33] that

$$SHI^3_*(Y(S),\alpha(S)) \cong KHI^3_*(Y,K;\pm 2g,0).$$

(This can be regarded as an instance of Theorem 5.16 on surface decompositions.)

To prove the existence of the desired representation, we can assume that $Y\backslash K$ is irreducible. In the case that $Y\backslash K$ is irreducible, $(Y(S),\alpha(S))$ is a taut sutured manifold. Corollary 5.19 implies that $SHI^3_*(Y(S),\alpha(S))$ is non-trivial and hence the rank of $KHI^3_*(Y,K)$ is at least 2. Now the claim follows from [DX20, Corollary 5.32]. $\square$

6 The Structure Theorem

In this section, we prove Theorem 1.10, the $U(3)$ analogue of Kronheimer and Mrowka's celebrated structure theorem for $U(2)$ Donaldson invariants [KM95]. In the first subsection, we provide background on Fukaya–Floer instanton homology, focusing on the case of $U(3)$. In the second subsection, using these preliminaries, we prove the structure theorem.

6.1 Fukaya–Floer homology of $S^1\times\Sigma_g$

Fukaya–Floer homology is a variation of instanton Floer homology that is helpful to understand the $U(N)$ Donaldson invariants of a pair (X,w) for some $z\in\mathbf{A}^N(X)$, where (X,w) is naturally written as a connected sum of (W,c) and (W',c') whose boundaries are an N-admissible pair (Y,γ) (with different orientations) but z is not necessarily induced by an element of $\mathbf{A}^N(W)\otimes\mathbf{A}^N(W')$ (see the gluing formula (6.5) below). The original idea of Fukaya–Floer homology goes back to [Fuk92], which was further developed in [BD95] in the case that $N=2$. Here we follow [DX20] to give a review of the general properties of Fukaya–Floer homology in the case that $N=3$, and hence we often drop "3" from our notations. Then we proceed to study Fukaya–Floer homology of $S^1\times\Sigma_g$. For more details on the background material, the reader can see Subsections 3.3 and 6.3 of [DX20]. We remark that even in the case $N=2$, the algebraic formulation of [DX20] is more involved than what is proposed in [BD95] because of bubbling phenomena.

Suppose (Y, γ) is an admissible pair and $L = (l_2, l_3)$ is a pair of elements of $H_1(Y; \mathbf{Z})$. The Fukaya–Floer homology group $\mathbb{I}_*(Y, \gamma, L)$ is a module over a ring R_3, which is defined in the following way. First for any non-negative integer j consider the ring

$$R_{3,j} := \mathbf{C}[s_{2,i}, s_{3,i}; 1 \leqslant i \leqslant j]/(s_{2,i}^2, s_{3,i}^2). \tag{6.1}$$

If $j \geqslant l$, then there is a homomorphism $R_{3,j} \to R_{3,l}$ that maps $s_{k,i}$ to $s_{k,i}$ if $i \leqslant l$ and to 0 if $i > l$. Now let R_3 be the inverse limit of this inverse system of rings. In particular,

$$t_k := \sum_{i=0}^{\infty} s_{k,i}$$

is an element of R_3 and this determines an algebra monomorphism from $\mathbf{C}[\![t_2, t_3]\!]$ to R_3.

The R_3-module $\mathbb{I}_*(Y, \gamma, L)$ is also defined as the inverse limit of an inverse system. For each j, there is a chain complex $(\mathfrak{C}_*^{\pi_j}(Y, \gamma) \otimes R_{3,j}, d_j)$ defined over the ring $R_{3,j}$, where $\mathfrak{C}_*^{\pi_j}(Y, \gamma)$ is a choice of instanton Floer chain complex for the admissible pair (Y, γ) and does not depend on L. The differential d_j has the form

$$d_j = \sum_{S_2,\, S_3 \subset [j]} \left(\prod_{i \in S_2} s_{2,i}\right)\left(\prod_{i \in S_3} s_{3,i}\right) d_j^{S_2, S_3} \tag{6.2}$$

where $[j] = \{1, 2, \ldots, j\}$ and $d_j^{S_2,S_3} : \mathfrak{C}_*^{\pi_j}(Y, \gamma) \to \mathfrak{C}_*^{\pi_j}(Y, \gamma)$. In particular, $d_j^{\varnothing,\varnothing}$ is the ordinary Floer differential. If $j \geqslant l$, then there is a chain map

$$F_{j,l} : (\mathfrak{C}_*^{\pi_j}(Y, \gamma) \otimes R_{3,j}, d_j) \to (\mathfrak{C}_*^{\pi_l}(Y, \gamma) \otimes R_{3,l}, d_l) \tag{6.3}$$

of $R_{3,j}$-modules such that $F_{l,k} \circ F_{j,l}$ is chain homotopy equivalent to $F_{j,k}$. Analogous to the differential maps d_j, the chain maps have the form

$$F_{j,l} = \sum_{S_2,\, S_3 \subset [j]} \left(\prod_{i \in S_2} s_{2,i}\right)\left(\prod_{i \in S_3} s_{3,i}\right) F_{j,l}^{S_2, S_3}, \tag{6.4}$$

where $F_{j,l}^{\varnothing,\varnothing} : \mathfrak{C}_*^{\pi_j}(Y, \gamma) \to \mathfrak{C}_*^{\pi_l}(Y, \gamma)$ is the continuation map defining a chain homotopy equivalence between two chain complexes representing $I_*(Y, \gamma)$. The homology of $(\mathfrak{C}_*^{\pi_j}(Y, \gamma) \otimes R_{3,j}, d_j)$ together with the homomorphisms induced by $F_{j,l}$ defines an inverse system and $\mathbb{I}_*(Y, \gamma, L)$ is the inverse limit of this system.

From (6.2) and (6.4), it is clear that the homomorphisms d_j and $F_{j,l}$ are compatible with a filtration on the Fukaya–Floer complexes. First define a filtration on $R_{3,j}$:

$$R_{3,j} = \mathcal{F}^0 R_{3,j} \supset \mathcal{F}^1 R_{3,j} \supset \mathcal{F}^2 R_{3,j} \cdots \supset \mathcal{F}^{2j+1} R_{3,j} = 0,$$

where $\mathcal{F}^k R_{3,j}$ contains linear combinations of monomials

$$\left(\prod_{i \in S_2} s_{2,i}\right)\left(\prod_{i \in S_3} s_{3,i}\right) \qquad \text{such that} \qquad |S_1| + |S_2| \geqslant k.$$

In particular, $\mathcal{F}^k R_{3,j} \cdot \mathcal{F}^l R_{3,j}$ is a subset of $\mathcal{F}^{k+l} R_{3,j}$, and the associated graded part of this filtration is a direct sum of 2^{2j} copies of $\mathbf{C}$. This filtration induces a filtration on $\mathfrak{C}_*^{\pi_j}(Y,\gamma) \otimes R_{3,j}$, and d_j is a filtration preserving homomorphism such that the induced map at the level of the associated graded part is $d_j^{\varnothing,\varnothing} \otimes 1$. A similar comment applies to $F_{j,l}$. From these filtrations one can obtain a spectral sequence for any j whose second page is $I_*(Y,\gamma) \otimes \mathbf{C}^{2^{2j}}$ and it abuts to $\mathbb{I}_*^{3,j}(Y,\gamma,L) := H(\mathfrak{C}_*^{\pi_j}(Y,\gamma) \otimes R_{3,j}, d_j)$.

Fukaya–Floer homology is functorial with respect to cobordisms. Suppose $(W,c) : (Y,\gamma) \to (Y',\gamma')$ is a cobordism of 3-admissible pairs, $z \in \mathbf{A}^3(W)$, and Γ, Λ are properly embedded oriented surfaces such that $\Gamma \cap Y$, $\Lambda \cap Y$ represent homology classes l_2, $l_3 \in H_1(Y;\mathbf{Z})$ and $\Gamma \cap Y'$, $\Lambda \cap Y'$ represent homology classes l_2', $l_3' \in H_1(Y';\mathbf{Z})$. Then there is an R_3-module homomorphism

$$\mathbb{I}(W,c,ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) : \mathbb{I}_*(Y,\gamma,L) \to \mathbb{I}_*(Y',\gamma',L')$$

with $L = (l_2,l_3)$ and $L' = (l_2',l_3')$. There is a slight variation of the above construction when one of the ends of the cobordism (W,c) is empty. If Y' is empty, then W is a 4-manifold with boundary $-Y$ and we have an R_3-module map

$$D_{W,c}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) : \mathbb{I}_*(Y,\gamma,L) \to R_3$$

and if Y is empty, then W is a 4-manifold with boundary Y' and we have an element

$$D_{W,c}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) \in \mathbb{I}_*(Y',\gamma',L').$$

These cobordism maps are defined by first constructing $R_{3,j}$-module chain maps between $(\mathfrak{C}_*^{\pi_j}(Y,\gamma) \otimes R_{3,j}, d_j)$ and $(\mathfrak{C}_*^{\pi_j}(Y',\gamma') \otimes R_{3,j}, d_j)$ that commute with the maps (6.3) up to chain homotopy. Furthermore, these chain maps respect the filtrations induced by that of $R_{3,j}$, and the leading order terms with respect to such filtrations are given by the cobordism maps of ordinary instanton Floer complexes.

The above homomorphisms are well-behaved with respect to composition of cobordisms. For instance if (W,c) is a pair with boundary (Y,γ) and (W',c') is a pair with boundary the orientation-reversal of (Y,γ), then we can glue them to obtain a closed pair $(W\#W', c\#c')$. If Γ, Λ are properly embedded surfaces in W and Γ', Λ' are properly embedded surfaces in W' such that Γ and Γ' (respectively, Λ and Λ') agree over the boundary and we can glue them to obtain a closed oriented embedded surface $\Gamma\#\Gamma'$ (respectively, $\Lambda\#\Lambda'$), then

$$\begin{aligned}\langle D_{W,c}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}),\ D_{W',c'}(z'e^{t_2\Gamma'_{(2)}+t_3\Lambda'_{(3)}})\rangle \\ = D_{W\#W',c\#c'}(zz'e^{t_2\Gamma\#\Gamma'_{(2)}+t_3\Lambda\#\Lambda'_{(3)}}),\end{aligned} \tag{6.5}$$

where $z \in \mathbf{A}^3(W)$ and $z' \in \mathbf{A}^3(W')$, and the left hand side is the obvious pairing. The invariant on the right hand side of (6.5) is given by $U(3)$ invariants of $(W\#W', c\#c')$ when $b^+(W\#W') > 1$. In the special case that $b^+(W\#W') = 1$, one can still define $U(3)$ polynomial invariants for $(W\#W', c\#c')$. However, this invariant depends on the choice of the metric and the right hand side of (6.5) is the invariant for a metric that we stretch along the embedded surface Y in $W\#W'$. We also remark that the right hand side of (6.5) is an element of the subalgebra $\mathbf{C}[\![t_2,t_3]\!]$ of R_3.

The main instance of $U(3)$ Fukaya–Floer homology relevant to this paper is that of $(S^1\times\Sigma_g,\gamma_d,L)$ with d coprime to 3 and $L=([S^1\times\{pt\}],[S^1\times\{pt\}])$. Following a similar notation as in Section 2, we write

$$\widetilde{\mathbb{V}}^3_{g,d}:=\mathbb{I}_*(S^1\times\Sigma_g,\gamma_d,L),$$

which is an R_3-module. Similar to (2.8), $[0,1]\times(S^1\times\Sigma_g,\gamma_d,L)$, viewed as a cobordism from two copies of $(S^1\times\Sigma_g,\gamma_d,L)$ to the empty set, induces a bilinear pairing

$$\langle\cdot,\cdot\rangle:\widetilde{\mathbb{V}}^3_{g,d}\otimes\widetilde{\mathbb{V}}^3_{g,d}\to R_3. \tag{6.6}$$

Analogous to $V^3_{g,d}$ and in the same way as in (2.16), $\widetilde{\mathbb{V}}^3_{g,d}$ is isomorphic to the cohomology of $\mathcal{N}_g$ with an appropriate coefficient ring. An explicit version of the isomorphism in (2.16) is given in [DX20, Theorem 3.18]. Focusing on the $N=3$ case, there is a vector space homomorphism $S:H^*(\mathcal{N}_g;\mathbf{C})\to\mathbf{A}^3_g$ such that the map

$$P:H^*(\mathcal{N}_g;\mathbf{C})[\varepsilon]/(\varepsilon^3-1)\to V^3_{g,d} \tag{6.7}$$

defined using the relative invariants

$$P(\varepsilon^i\cdot\sigma)=D_{\Delta_g,\delta_{g,d}+i\Sigma}(S(\sigma)),$$

with $\Delta_g:=D^2\times\Sigma_g$ and $\delta_{g,d}=D^2\times\{x_1,\dots,x_d\}$, is an isomorphism. Similarly, we can define $\mathbb{P}:H^*(\mathcal{N}_g;R_3)[\varepsilon]/(\varepsilon^3-1)\to\widetilde{\mathbb{V}}^3_{g,d}$, an analogue of (6.7) for the Fukaya–Floer homology group of $S^1\times\Sigma_g$, by setting

$$\mathbb{P}(\varepsilon^i\cdot\sigma)=D_{\Delta_g,\delta_{g,d}+i\Sigma}(S(\sigma)e^{t_2D_{(2)}+t_3D_{(3)}}). \tag{6.8}$$

Here we extend S as a module homomorphism $H^*(\mathcal{N}_g;\mathbf{C})\otimes R_3\to\mathbf{A}^3_g\otimes R_3$ in the obvious way and D denotes the disc $D\times\{pt\}$ in Δ_g.

Lemma 6.9. *The R_3-module map $\mathbb{P}:H^*(\mathcal{N}_g;R_3)[\varepsilon]/(\varepsilon^3-1)\to\widetilde{\mathbb{V}}^3_{g,d}$ is an isomorphism. In particular, $\widetilde{\mathbb{V}}^3_{g,d}$ is a free R_3-module.*

Proof. It suffices to show that $\mathbb{P}_j:H^*(\mathcal{N}_g;R_{3,j})[\varepsilon]/(\varepsilon^3-1)\to\mathbb{I}^{3,j}_*(S^1\times\Sigma_g,\gamma_{g,d},L_g)$, the $R_{3,j}$-module homomorphism given by

$$\mathbb{P}_j(\varepsilon^i\cdot\sigma)=D^{3,j}_{\Delta_g,\delta_{g,d}+i\Sigma}(S(\sigma)e^{D_{(2)}+D_{(3)}}),$$

is an isomorphism. Since $\mathbb{P}_j$ is an $R_{3,j}$-module homomorphism, it is a filtration preserving homomorphism with respect to the filtration induced by that of $R_{3,j}$. The induced morphism of spectral sequences on the second page maps $H^*(\mathcal{N}_g;\mathbf{C})[\varepsilon]/(\varepsilon^3-1)\otimes R_{3,j}$ to $I_*(S^1\times\Sigma_g,\gamma_{g,d})\otimes R_{3,j}$ by the map $P\otimes 1$. In particular, it is an isomorphism. □

The ring R_3 is not an integral domain and for our purposes it is easier to work with modules over an integral domain. We define $\mathbb{V}^3_{g,d}$ be the $\mathbf{C}[\![t_2,t_3]\!]$-module given by

$$\mathbb{V}^3_{g,d} := \text{im}\left(\mathbb{P}|_{H^*(\mathcal{N}_g;\mathbf{C}[\![t_2,t_3]\!])[\varepsilon]/(\varepsilon^3-1)}\right)$$

It is clear from the definition that $\mathbb{V}^3_{g,d}$ as a $\mathbf{C}[\![t_2,t_3]\!]$-module is isomorphic to $\mathbf{C}[\![t_2,t_3]\!]^{3n_g}$ with $n_g = \dim_{\mathbf{C}} H^*(\mathcal{N}_g;\mathbf{C})$. We have the following alternative identification.

Lemma 6.10. *Suppose $\{\sigma_k\}_{1\leqslant k\leqslant n_g}$ is a basis for $H^*(\mathcal{N}_g;\mathbf{C})$ as a vector space over $\mathbf{C}$. Then, the map $\Phi : \mathbb{V}^3_{g,d} \to \mathbf{C}[\![t_2,t_3]\!]^{3n_g}$ given by*

$$\Phi(\zeta) := (\langle\langle \zeta, D_{\Delta_g,\delta_{g,d}+i\Sigma}(S(\sigma_k)e^{t_2D_{(2)}+t_3D_{(3)}})\rangle\rangle)_{1\leqslant k\leqslant n_g,\,0\leqslant i\leqslant 2}, \tag{6.11}$$

defined using the pairing (6.6)*, is an isomorphism.*

Note that (6.5) implies the right side of (6.11) is indeed an element of $\mathbf{C}[\![t_2,t_3]\!]^{3n_g}$.

Proof. If ζ is in the kernel of Φ, then its pairing with any element of $\widetilde{\mathbb{V}}^3_{g,d}$ vanishes. Thus [DX20, Proposition 3.30] implies that $\zeta = 0$. It remains to show that Φ is surjective. Fix

$$\mathbf{v} = \sum_{k=0}^{\infty}\sum_{i=0}^{k} t_2^i t_3^{k-i} v_{i,k-i} \in \mathbf{C}[\![t_2,t_3]\!]^{3n_g},$$

where $v_{i,j} \in \mathbf{C}^{3n_g}$. We inductively define an element

$$\sigma = \sum_{k=0}^{\infty}\sum_{i=0}^{k} t_2^i t_3^{k-i}\sigma_{i,k-i}$$

with $\sigma_{i,j} \in H^*(\mathcal{N}_g;\mathbf{C})[\varepsilon]/(\varepsilon^3-1)$ such that for any integer n, in the expression

$$\Phi\circ\mathbb{P}(\sum_{k=0}^{n}\sum_{i=0}^{k} t_2^i t_3^{k-i}\sigma_{i,k-i}) - \mathbf{v} \in \mathbf{C}[\![t_2,t_3]\!]^{3n_g}$$

only terms of the form $t_2^i t_3^j$ with $i + j > n$ appear. In fact, assuming this holds for a given n, then we have the following, where $w_{i,j} \in \mathbf{C}^{3n_g}$:

$$\Phi\circ\mathbb{P}(\sum_{k=0}^{n}\sum_{i=0}^{k} t_2^i t_3^{k-i}\sigma_{i,k-i}) - \mathbf{v} = \sum_{k=n+1}^{\infty}\sum_{i=0}^{k} t_2^i t_3^{k-i} w_{i,k-i}$$

The non-degeneracy of the pairing on $V^3_{g,d}$ implies that for any $0 \leqslant i \leqslant n+1$, there is a unique $\sigma_{i,n+1-i} \in H^*(\mathcal{N}_g;\mathbf{C})[\varepsilon]/(\varepsilon^3-1)$ such that $\phi\circ P(\sigma_{i,n+1-i}) = w_{i,n+1-i}$. Here $\phi : V^3_{g,d} \to \mathbf{C}^{3n_g}$ is defined in a similar way as Φ. It is straightforward to check that we can carry out the induction step with this choice of $\sigma_{i,j}$ when $i + j = n + 1$. $\square$

Corollary 6.12. *Any relative invariant $D_{\Delta_g,\delta_{g,d}+i\Sigma}(ze^{D_{(2)}+D_{(3)}})$ where $z \in \mathbf{A}_g^3$ is an element of $\mathbb{V}_{g,d}^3$. In particular, $\mathbb{V}_{g,d}^3$ is the $\mathbf{C}[\![t_2,t_3]\!]$-module generated by such invariants.*

Proof. Using Lemma 6.10, there is $\zeta \in \mathbb{V}_{g,d}^3$ such that

$$\langle \zeta - D_{\Delta_g,\delta_{g,d}+j\Sigma}(ze^{D_{(2)}+D_{(3)}}),\ \ D_{\Delta_g,\delta_{g,d}+i\Sigma}(S(\sigma)e^{t_2D_{(2)}+t_3D_{(3)}})\rangle = 0.$$

This implies that the pairing of $\zeta - D_{\Delta_g,\delta_{g,d}+j\Sigma}(ze^{D_{(2)}+D_{(3)}})$ with any element of $\widetilde{\mathbb{V}}_{g,d}^3$ is trivial, and hence by [DX20, Proposition 3.30] this element vanishes. In particular, $\zeta = D_{\Delta_g,\delta_{g,d}+j\Sigma}(ze^{D_{(2)}+D_{(3)}})$ belongs to $\mathbf{C}[\![t_2,t_3]\!]$. □

For any integer i and $z \in \mathbf{A}_g^3$, consider the homomorphism

$$\mathbb{I}([-1,1]\times S^1\times\Sigma_g, [-1,1]\times\gamma_{g,d}+i\Sigma_g, ze^{t_2C_{(2)}+t_3C_{(3)}}) : \widetilde{\mathbb{V}}_{g,d}^3 \to \widetilde{\mathbb{V}}_{g,d}^3$$

where $C = [-1,1]\times S^1\times\{pt\}$. Corollary 6.12 and functoriality of Fukaya–Floer homology implies that this homomorphism maps $\mathbb{V}_{g,d}^3$ to itself. This gives $\mathbb{V}_{g,d}^3$ the structure of a cyclic module over $\mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]/(\varepsilon^3-1)$. That is to say, there is an ideal $\mathbb{J}_{g,d}^3$ of $\mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]$ containing ε^3-1 such that

$$\mathbb{V}_{g,d}^3 = \mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]/\mathbb{J}_{g,d}^3.$$

The following is another consequence of Lemma 6.10.

Corollary 6.13. *For any element z of the ideal $J_{g,d}^3 \subset \mathbf{A}_g^3[\varepsilon]$, there is $\mathbf{z} \in \mathbb{J}_{g,d}^3$ such that*

$$\mathbf{z} = z + \sum_{k=1}^{\infty}\sum_{i=0}^{\infty} t_2^i t_3^{k-i} z_{i,k-i}.$$

That is to say, z is the constant term of the power series $\mathbf{z}$.

Proof. We may regard z as an element of $\mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]$ where the coefficient of $t_2^it_3^j$ is zero unless $i=j=0$. Thus z determines an element ζ of $\mathbb{V}_{g,d}^3$. Since $z \in J_{g,d}^3$, $\Phi(\zeta)$ has a trivial constant term and hence we can find $\mathbf{v}_2,\ \mathbf{v}_3 \in \mathbf{C}[\![t_2,t_3]\!]^{3n_g}$ such that

$$\Phi(\zeta) = t_2\mathbf{v}_2 + t_3\mathbf{v}_3. \tag{6.14}$$

By Lemma 6.10, we can find $\eta_2,\ \eta_3 \in \mathbb{V}_{g,d}^3$ such that $\Phi(\eta_i) = \mathbf{v}_i$. In particular, $\zeta - t_2\eta_2 - t_3\eta_3$ is a trivial element of $\mathbb{V}_{g,d}^3$. Since $\mathbb{V}_{g,d}^3$ is a cyclic module over $\mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]/(\varepsilon^3-1)$, there are $\mathbf{z}_2,\ \mathbf{z}_3 \in \mathbf{A}_g^3\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]/(\varepsilon^3-1)$ such that $\mathbf{z}_i$ is mapped to η_i. This implies that $\mathbf{z} := z - t_2\mathbf{z}_2 - t_3\mathbf{z}_3$ is in $\mathbb{J}_{g,d}^3$ and has the form in (6.14). □

Next, we define a Fukaya–Floer analogue of the simple-type ideal (2.17):

$$\mathbb{S}_{g,d}^3 := \ker(\beta_2^3-27)\cap\ker(\beta_3)\cap\bigcap_{\substack{r=2,3\\ 1\leqslant j\leqslant 2g}}\ker(\psi_r^j) \subset \mathbb{V}_{g,d}^3.$$

An important ingredient in the proof of the structure theorem involves an understanding of this $\mathbf{C}[\alpha_2,\alpha_3,\beta_2,\beta_3][\![t_2,t_3]\!]$-module. The following is an adaption of the main argument that proves Theorem 2.14, given in Section 2.

Theorem 6.15. *$\mathbb{S}^3_{g,d}$ is a free $\mathbf{C}[\![t_2,t_3]\!]$-module of rank $3(2g-1)^2$. Moreover,*

$$\mathbb{S}^3_{g,d} = \bigoplus_{\substack{k\in\{0,1,2\}\\(a,b)\in\mathcal{C}_g}} R_{k,a,b} \tag{6.16}$$

where $R_{k,a,b}$ is the free $\mathbf{C}[\![t_2,t_3]\!]$-module of rank one given by

$$R_{k,a,b} = \frac{\mathbf{C}[\![t_2,t_3]\!][\alpha_2,\alpha_3,\beta_2,\beta_3]}{(\alpha_2-(\zeta^k\sqrt{3}a+\zeta^{2k}t_2), \alpha_3-(\zeta^{2k}\sqrt{-3}b-2\zeta^k t_3), \beta_2-3\zeta^{2k}, \beta_3)} \tag{6.17}$$

The above description also determines $\mathbb{S}^3_{g,d}$ as an $\mathbf{C}[\alpha_2,\alpha_3,\beta_2,\beta_3][\![t_2,t_3]\!]$-module.

As preparation for the proof, we need the *blowup formula* for $U(3)$ polynomial invariants. This will also be an essential ingredient in the proof of the structure theorem. Write $\widehat{X}$ for a blowup of X, and denote the exceptional class by $E\in H^2(\widehat{X};\mathbf{Z})$. The following is essentially due to Culler [Cul14], and is stated in [DX20, §2.5].

Theorem 6.18. *If (X,w) is $U(3)$ simple type, then for $\Gamma,\Lambda\in H^2(X;\mathbf{Z})$, we have*

$$\mathbb{D}_{\widehat{X},w}(t_2E_{(2)}+t_3E_{(3)}+\Gamma_{(2)}+\Lambda_{(3)}) = \frac{1}{3}e^{-t_2^2/2+t_3^2}\left(\cosh(\sqrt{3}t_2)+2\cos(\sqrt{3}t_3)\right)\mathbb{D}_{X,w}(\Gamma_{(2)}+\Lambda_{(3)}),$$

$$\mathbb{D}_{\widehat{X},w+E}(t_2E_{(2)}+t_3E_{(3)}+\Gamma_{(2)}+\Lambda_{(3)}) = \frac{1}{3}e^{-t_2^2/2+t_3^2}\left(\cosh(\sqrt{3}t_2)-\cos(\sqrt{3}t_3)-\sqrt{3}\sin(\sqrt{3}t_3)\right)\mathbb{D}_{X,w}(\Gamma_{(2)}+\Lambda_{(3)}).$$

Below, we will make use of the identity

$$D_{X,w}((1+\frac{1}{3}\zeta^k x_{(2)}+\frac{1}{9}\zeta^{2k}x_{(2)}^2)e^z) = \zeta^{k\,d_w}\mathbb{D}_{X,w}(\zeta^{-k\deg(z)}z) \tag{6.19}$$

where $d_w := b^+(X)-b^1(X)-w\cdot w+1$ and z is a homogenous element of $\mathbf{A}^3(X)$. This relation follows from the observation that the mod 3 dimension of the moduli spaces of $U(3)$ instantons for (X,w) is fixed and equal to d_w.

Proof of Theorem 6.15. Suppose $\widetilde{\mathbb{J}}^3_{g,d}$ is the ideal of $\mathbf{A}^3_g\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]$ generated by $\mathbb{J}^3_{g,d}$ and $(\beta_2^3-27,\beta_3,\psi_2^i,\psi_3^i)_{i=1}^{2g}$. The pairing $\langle\cdot,\cdot\rangle : \mathbb{V}^3_{g,d}\otimes_{\mathbf{C}[\![t_2,t_3]\!]}\mathbb{V}^3_{g,d}\to\mathbf{C}[\![t_2,t_3]\!]$ induces

$$\mathbb{S}^3_{g,d}\otimes_{\mathbf{C}[\![t_2,t_3]\!]}\mathbb{V}^3_{g,d}/(\beta_2^3-27,\beta_3,\psi_2^i,\psi_3^i)_{i=1}^{2g}\to\mathbf{C}[\![t_2,t_3]\!].$$

The non-degeneracy of the pairing gives

$$\operatorname{rank}_{\mathbf{C}[\![t_2,t_3]\!]}(\mathbb{S}^3_{g,d}) \leqslant \operatorname{rank}_{\mathbf{C}[\![t_2,t_3]\!]}\left(\mathbf{A}^3_g\otimes\mathbf{C}[\![t_2,t_3]\!][\varepsilon]/\widetilde{\mathbb{J}}^3_{g,d}\right).$$

Corollary 6.13 implies that the right hand side of the above inequality is not greater than

$$\dim_{\mathbf{C}} \mathbf{A}^3_g[\varepsilon]/\widetilde{J}^3_{g,d} = 3(2g-1)^2,$$

where the latter is established in Section 2. Thus

$$\mathrm{rank}_{\mathbf{C}[\![t_2,t_3]\!]} \mathbb{S}^3_{g,d} \leqslant 3(2g-1)^2. \tag{6.20}$$

We can construct elements of the simple-type ideal using a $K3$ surface. (A similar construction can be done for smooth 4-manifolds of $U(3)$ simple type.) Let Σ' be a surface of genus g in a $K3$ surface with $\Sigma' \cdot \Sigma' = 2g-2$. For instance, we can construct Σ' in the following way. The 4-manifold $K3$ admits an elliptic fibration with a section that is a (-2)-embedded sphere. The union of this sphere and g regular fibers, after resolving the intersection points, gives a surface with the desired genus and self intersection number. We fix another surface F with $F \cdot F = 0$ and $F \cdot \Sigma' = 1$. For instance, take F to be a regular fiber. Let also w be the union of d other regular fibers and regard it as a 2-cycle in $K3$ with trivial self-intersection number. Next, let X be the blowup of the $K3$ surface at $2g-2$ points on Σ' away from F and w, and denote the proper transform of Σ' by Σ. Then Σ determines a surface of genus g and self intersection number 0. We also obtain a surface and a 2-cycle in X induced by F and w, which are denoted by the same notation.

Removing a regular neighborhood of Σ from X, w and F determines a 4-manifold X° with boundary $S^1 \times \Sigma_g$, a 2-cycle w° that intersects the boundary of X at $S^1 \times \{x_1, \dots, x_d\}$ and an embedded surface F° which intersects the boundary at $S^1 \times \{y\}$. In particular, for any $z \in \mathbf{A}^3(X)$, the following is an element of $\mathbb{V}^3_{g,d}$:

$$D_{X^\circ,w^\circ}(ze^{t_2F^\circ_{(2)}+t_3F^\circ_{(3)}}). \tag{6.21}$$

Analogous to (2.29), one can see the above element of $\mathbb{V}^3_{g,d}$ belongs to $\mathbb{S}^3_{g,d}$.

Next, we define a homomorphism $\Psi : \mathbb{S}^3_{g,d} \to \mathbf{C}[\![t_2,t_3]\!]^{3(2g-1)^2}$ and use the upper bound in (6.20) on the rank of $\mathbb{S}^3_{g,d}$ and the elements of $\mathbb{S}^3_{g,d}$ constructed in (6.21) to show that Ψ is an isomorphism. For any $\lambda = (a,b,k)$ in $\mathcal{C}_g \times \{0,1,2\}$, let $P_\lambda \in \mathbf{C}[\![t_2,t_3]\!][w,x,y]$ be a polynomial such that for any $\lambda' = (a',b',k') \in \mathcal{C}_g \times \{0,1,2\}$, the value of P_λ at

$$(3\zeta^{2k'}, \sqrt{3}\zeta^{k'}a' + \zeta^{2k'}t_2, \sqrt{-3}\zeta^{2k'}b' - 2\zeta^{k'}t_3)$$

is 1 if $\lambda' = \lambda$ and is 0 if $\lambda' \neq \lambda$. The homomorphism Ψ is defined as

$$\Psi(\zeta) := \left\{\langle \zeta, D_{\Delta_g,\delta_{g,d}}(P_\lambda(x_{(2)},\Sigma_{(2)},\Sigma_{(3)})e^{D_{(2)}+D_{(3)}})\rangle\right\}_\lambda.$$

To compute $\Psi(\zeta)$ for an element of $\mathbb{S}^3_{g,d}$ that is a relative invariant as in (6.21), we can use the pairing formula (6.5) to compute

$$\langle D_{X^\circ,w^\circ}(ze^{t_2F^\circ_{(2)}+t_3F^\circ_{(3)}}),\ \ D_{\Delta_g,\delta_{g,d}}(P_\lambda(x_{(2)},\Sigma_{(2)},\Sigma_{(3)})e^{D_{(2)}+D_{(3)}})\rangle$$
$$= D_{X,w}(zP_\lambda(x_{(2)},\Sigma_{(2)},\Sigma_{(3)})e^{t_2F_{(2)}+t_3F_{(3)}}). \tag{6.22}$$

We consider the special case that

$$z = (1 + \frac{1}{3}\zeta^k x_{(2)} + \frac{1}{9}\zeta^{2k} x_{(2)}^2) P(x_{(2)}, \Sigma_{(2)}, \Sigma_{(3)}) \tag{6.23}$$

for some $P \in \mathbf{C}[\![t_2, t_3]\!][w, x, y]$. Then the right hand side of (6.22) is given by evaluating the following expression at $s_2 = s_3 = 0$:

$$R(3\zeta^{2k}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}) \widehat{D}^{\zeta^k}_{X,w}(e^{s_2\Sigma_{(2)} + s_3\Sigma_{(3)} + t_2 F_{(2)} + t_3 F_{(3)}}) \tag{6.24}$$

where R is the element of $\mathbf{C}[\![t_2, t_3]\!][w, x, y]$ given by $P \cdot P_\lambda$. Here we use the fact that $K3$ has $U(3)$ simple type [DX20]. For any cycle w in a $K3$ surface and Γ, $\Lambda \in H_2(K3)$, the $U(3)$ Donaldson-type invariant is computed in [DX20] to be

$$\mathbb{D}_{K3,w}(\Gamma_{(2)} + \Lambda_{(3)}) = e^{\frac{Q(\Gamma)}{2} - Q(\Lambda)}. \tag{6.25}$$

This identity, Theorem 6.18 and (6.19) can be used to show that (6.24) is equal to

$$R(3\zeta^{2k}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}) \left[\frac{1}{3^{2g-2}} \zeta^k e^{\zeta^{2k} s_2 t_2 - 2\zeta^k s_3 t_3} (\cosh(\sqrt{3}\zeta^k s_2) + 2\cos(\sqrt{3}\zeta^{2k} s_3))^{2g-2} \right].$$

In particular, (6.24) is equal to

$$R(3\zeta^{2k}, \frac{\partial}{\partial s_2}, \frac{\partial}{\partial s_3}) \left[e^{\zeta^{2k} s_2 t_2 - 2\zeta^k s_3 t_3} \sum_{(a,b) \in \mathcal{C}_g} c_{a,b} e^{\sqrt{3}\zeta^k a s_2 + \sqrt{-3}\zeta^{2k} b s_3} \right],$$

for some non-zero constants $c_{a,b}$. We may simplify the above expression as

$$e^{\zeta^{2k} s_2 t_2 - 2\zeta^k s_3 t_3} \sum_{(a,b) \in \mathcal{C}_g} c_{a,b} R(3\zeta^{2k}, \sqrt{3}\zeta^k a + \zeta^{2k} t_2, \sqrt{-3}\zeta^{2k} b - 2\zeta^k t_3) e^{\sqrt{3}\zeta^k a s_2 + \sqrt{-3}\zeta^{2k} b s_3}.$$

The assumption $R = P \cdot P_\lambda$ for a fixed $\lambda = (a, b)$ can be used to further (6.24) simplify as

$$c_{a,b} e^{\zeta^{2k} s_2 t_2 - 2\zeta^k s_3 t_3} P(3\zeta^{2k}, \sqrt{3}\zeta^k a + \zeta^{2k} t_2, \sqrt{-3}\zeta^{2k} b - 2\zeta^k t_3) e^{\sqrt{3}\zeta^k a s_2 + \sqrt{-3}\zeta^{2k} b s_3}.$$

For a given $(a_0, b_0, k_0) \in \mathcal{C}_g \times \{0, 1, 2\}$, we pick z in (6.23) with $k = k_0$ and $P = P_{\lambda_0}$ where $\lambda_0 = (a_0, b_0)$. Then all components of Ψ applied to this element of $\mathbb{S}^3_{g,d}$ are equal to 0 except the component corresponding to (a_0, b_0, k_0), which is a non-zero real number. This shows that the map Ψ is surjective. This observation and (6.20) imply that the rank of $\mathbb{S}^3_{g,d}$ is $3(2g-1)^2$ and the kernel of Ψ is torsion. However, the kernel is a submodule of the free module $\mathbb{V}^3_{g,d}$, and hence the kernel of Ψ is trivial. Consequently, Ψ gives an isomorphism, and is the direct sum of rank 1 modules given by the above elements as (a_0, b_0, k_0) ranges over all elements of $\mathcal{C}_g \times \{0, 1, 2\}$. Furthermore, the computation of the previous paragraph shows that any such rank 1 summand is invariant with respect to the action of the operators α_2, α_3, β_2 and β_3, and it is isomorphic R_{k_0,a_0,b_0}, defined as in (6.17). □

6.2 Proof of the structure theorem

We now prove the structure theorem. For the convenience of the reader, we recall the statement of Theorem 1.10. Retaining the convention of the previous subsection, we write $D_{X,w}$ for $D^3_{X,w}$, and so forth. Let $\zeta = e^{2\pi i/3}$.

Theorem 6.26. *Suppose $b^+(X) > 1$, and X is $U(3)$ simple type. Then there is a finite set $\{K_i\} \subset H^2(X;\mathbf{Z})$ and $c_{i,j} \in \mathbf{Q}[\sqrt{3}]$ such that for any $w \in H^2(X;\mathbf{Z})$, and $\Gamma, \Lambda \in H_2(X)$:*

$$\mathbb{D}_{X,w}(\Gamma_{(2)} + \Lambda_{(3)}) = e^{\frac{Q(\Gamma)}{2} - Q(\Lambda)} \sum_{i,j} c_{i,j} \zeta^{w\cdot\left(\frac{K_i - K_j}{2}\right)} e^{\frac{\sqrt{3}}{2}(K_i+K_j)\cdot\Gamma + \frac{\sqrt{-3}}{2}(K_i - K_j)\cdot\Lambda}$$

Each class K_i is an integral lift of $w_2(X)$, and satisfies the following: if $\Sigma \subset X$ is a smoothly embedded surface of genus g with $\Sigma\cdot\Sigma \geqslant 0$ and $[\Sigma]$ non-torsion, then

$$2g - 2 \geqslant |\langle K_i, \Sigma\rangle| + [\Sigma]^2. \tag{6.27}$$

Remark 6.28. The authors suspect a stronger statement holds: namely, that if (X, w) is $U(3)$ simple type for any single $w \in H^2(X;\mathbf{Z})$, then X is $U(3)$ simple type. However, it appears that to adapt the proof to address such a statement requires equality in Proposition 2.11, and does not follow from the partial description of eigenvalues given in Theorem 2.14.

Remark 6.29. Kronheimer and Mrowka used an adjunction inequality in the $U(2)$ setting [KM93] to prove the Milnor conjecture on the slice genus of torus knots. This motivated the introduction of concordance invariants constructed from versions of $U(2)$ instanton Floer theory for knots [KM11b, KM13]. The $U(3)$ adjunction inequality in Theorem 6.26 implies the Milnor conjecture in a similar way, and it would be interesting to explore whether there are similar concordance invariants that can be defined using $U(3)$ instantons.

Our proof of Theorem 6.26 is largely an adaptation of Muñoz's proof in the $N = 2$ case [Muñ00], which uses $U(2)$ Fukaya–Floer homology. A key ingredient is Theorem 6.15, regarding the $U(3)$ Fukaya–Floer analogue of the simple-type ideal (2.17). We being with the following $N = 3$ analogue of Lemma 11 from [Muñ00].

Lemma 6.30. *Suppose X satisfies $b^+(X) > 1$ and is $U(3)$ simple type. Fix $w \in H^2(X;\mathbf{Z})$. Let $\Sigma \subset X$ be a surface of genus g with $[\Sigma]^2 = 0$ and $\Sigma \cdot w = d \not\equiv 0 \pmod 3$. Then there are $h_{a,b} \in \mathbf{C}[\![t_2, t_3]\!]$ such that for all $\Gamma, \Lambda \in H_2(X)$ and $l \in \mathbf{Z}$, we have:*

$$\mathbb{D}_{X,w+l\Sigma}(s_2\Sigma_{(2)} + s_3\Sigma_{(3)} + t_2\Gamma_{(2)} + t_3\Lambda_{(3)}) \tag{6.31}$$

$$= e^{Q(s_2\Sigma + t_2\Gamma)/2 - Q(s_3\Sigma + t_3\Gamma)} \sum_{(a,b)\in\mathcal{C}_g} \zeta^{lb} h_{a,b} e^{\sqrt{3}as_2 + \sqrt{-3}bs_3}$$

Proof. We may suppose Γ and Λ are represented by surfaces which intersect Σ transversely in a single point, and $\Gamma\cdot\Sigma = \Lambda\cdot\Sigma = 1$. The general case follows from this case and linearity of the resulting expression. Identify a regular neighborhood of $\Sigma \subset X$ with $D^2 \times \Sigma$, and write $X = X^\circ \cup D^2 \times \Sigma$. Write $\Gamma = \Gamma^\circ \cup D$ and similarly for Λ, where D and δ both denote $D^2 \times \{pt\}$ in $\Delta = D^2 \times \Sigma$. As in (6.22), the gluing formula (6.5) gives

$$D_{X,w}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) = \langle D_{X^\circ,w^\circ}(ze^{t_2\Gamma^\circ_{(2)}+t_3\Lambda^\circ_{(3)}}), D_{\Delta,\delta}(e^{t_2D_{(2)}+t_3D_{(3)}})\rangle \tag{6.32}$$

for all $z \in \mathbf{A}^3(X,\Sigma)$, where we define $[\Sigma]^\perp = \{y \in H_2(X) | y \cdot \Sigma = 0\}$, and

$$\mathbf{A}^3(X,\Sigma) := \big(\mathrm{Sym}^*(H_0(X) \oplus [\Sigma]^\perp) \otimes \Lambda^* H_1(X)\big)^{\otimes 2} \subset \mathbf{A}^3(X).$$

The two invariants appearing on the right side of (6.32) are elements of the Fukaya–Floer homology $\mathbb{V}^3_{g,d}$. Now let s_2, s_3 be formal variables and set

$$z = (1 + \frac{1}{3}x_{(2)} + \frac{1}{9}x_{(2)}^2)e^{s_2\Sigma_{(2)}+s_3\Sigma_{(3)}}. \tag{6.33}$$

Then the left side of (6.32) is equal to the left side of (6.31) when $l = 0$. By the simple type assumption and the gluing formula, we have

$$D_{X^\circ,w^\circ}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) \in \mathbb{S}^3_{g,d} \otimes_{\mathbf{C}} \mathbf{C}[\![s_2,s_3]\!].$$

Furthermore, by Theorem 6.15 we can write

$$D_{X^\circ,w^\circ}(ze^{t_2\Gamma_{(2)}+t_3\Lambda_{(3)}}) = \sum_{(a,b)\in\mathcal{C}_g} f^w_{a,b}$$

where $f^w_{a,b} \in R_{0,a,b} \otimes_{\mathbf{C}} \mathbf{C}[\![s_2,s_3]\!]$. (The presence of $1 + x_{(2)}/3 + x_{(2)}^2/9$ in z implies $k = 0$ in (6.16).) From the description of $R_{0,a,b}$, $f^w_{a,b}$ is a solution of the differential operator

$$\left(\frac{\partial}{s_2} - (\sqrt{3}a + t_2)\right)\left(\frac{\partial}{s_3} - (\sqrt{-3}b - 2t_3)\right). \tag{6.34}$$

By the gluing formula we can then write

$$\mathbb{D}_{X,w}(s_2\Sigma_{(2)} + s_3\Sigma_{(3)} + t_2\Gamma_{(2)} + t_3\Lambda_{(3)}) = \sum_{(a,b)\in\mathcal{C}_g} g^w_{a,b}$$

where $g^w_{a,b} \in \mathbf{C}[\![s_2,s_3,t_2,t_3]\!]$ is given by the pairing $\langle f^w_{a,b}, D_{\Delta,\delta}(e^{t_2D_{(2)}+t_3D_{(3)}})\rangle$. Furthermore, $g^w_{a,b}$ is also a solution of the operator (6.34). Thus we obtain

$$g^w_{a,b} = h^w_{a,b}(t_2,t_3)e^{\sqrt{3}as_2+s_2t_2+\sqrt{-3}bs_3-2s_3t_3}.$$

This proves the claim in the case $l = 0$.

For the case of general l, first note the above argument carries through to show that

$$\mathbb{D}_{X,w+l\Sigma}(s_2\Sigma_{(2)} + s_3\Sigma_{(3)} + t_2\Gamma_{(2)} + t_3\Lambda_{(3)}) = \sum_{(a,b)\in\mathcal{C}_g} h^{w+l\Sigma}_{a,b}e^{\sqrt{3}as_2+s_2t_2+\sqrt{-3}bs_3-2s_3t_3}$$

for some $h_{a,b}^{w+l\Sigma} \in \mathbf{C}[\![t_2, t_3]\!]$. Next, recall that there is a class $\varepsilon = \varepsilon(\Sigma)$ that acts on $\mathbb{V}_{g,d}^3$ as an operator of degree $-4d \pmod{4N}$, and can also be used via the gluing formula as a class when evaluating $D_{X,w}$. Namely, in the situation at hand, we have the relation

$$D_{X,w}(\varepsilon^l z) = D_{X,w+l\Sigma}(z)$$

for any $z \in \mathbf{A}^3(X, \Sigma)$. The operator ε restricted to $\mathbb{S}_{g,d}^3$ acts as follows:

$$\varepsilon : R_{k,a,b} \to R_{k,a,b} \text{ is multiplication by } \zeta^{b+dk}.$$

This follows from the fact that the eigenvalues in (2.12) have $\varepsilon = +1$, combined with the argument of Lemma 2.5 (using that ε has degree $-4d \pmod{4N}$). Replace z in (6.33) by

$$z = (1 + \zeta^i\varepsilon + \zeta^{2i}\varepsilon^2)(1 + \frac{1}{3}x_{(2)} + \frac{1}{9}x_{(2)}^2)e^{s_2\Sigma_{(2)}+s_3\Sigma_{(3)}}.$$

From this substitution, carrying the above argument through, we obtain the relation

$$\sum_{l=0,1,2} \zeta^{li} h_{a,b}^{w+l\Sigma} = 0 \quad \text{if} \quad b + i \not\equiv 0 \pmod 3 \tag{6.35}$$

for each $i \in \mathbf{Z}$; the key point is that the term $(1 + \zeta^i\varepsilon + \zeta^{2i}\varepsilon^2)$ places the relative invariants in the (ζ^{-i})-eigenspace of ε. The relations (6.35) are then used to solve

$$h_{a,b}^{w+l\Sigma} = \zeta^{lb} h_{a,b}^w.$$

This proves the claimed formula for general l, upon setting $h_{a,b} := h_{a,b}^w$. □

Proof of Theorem 6.26. The proof runs parallel to Steps 2–5 of the proof of Theorem 2 from [Muñ00] for the $N = 2$ case; we begin with an analogue of Step 2. Fix X a smooth closed oriented 4-manifold with $b^+(X) > 1$ and of $U(3)$ simple type. We first show that for each $w \in H^2(X;\mathbf{Z})$ there exists a finite set $\{K_i\}_{i\in I} \subset H^2(X;\mathbf{Z})$ and $c_{i,j}^w \in \mathbf{Q}[\sqrt{3}]$ such that

$$\mathbb{D}_{X,w}(\Gamma_{(2)} + \Lambda_{(3)}) = e^{\frac{Q(\Gamma)}{2} - Q(\Lambda)} \sum_{i,j} c_{i,j}^w e^{\frac{\sqrt{3}}{2}(K_i+K_j)\cdot\Gamma + \frac{\sqrt{-3}}{2}(K_i-K_j)\cdot\Lambda} \tag{6.36}$$

The blow-up formulas of Theorem 6.18 show that it suffices to prove this for any blowup of X. For simplicity we assume $H_2(X;\mathbf{Z})$ has no torsion (in the general case, mod out by torsion in the argument). After possibly blowing up, we may assume (using $b^+(X) > 1$) that the intersection form of X can be diagonalized, $Q = (+1)^r \oplus (-1)^s$. Let $A_1, \ldots, A_r, B_1, \ldots, B_s$ be a corresponding basis, so that $A_k^2 = 1$, $B_k^2 = -1$, $A_k \cdot B_k = 0$. Define

$$\Sigma^1 = A_2 - B_1, \qquad \Sigma^k = -A_k - B_1 \quad (2 \leqslant k \leqslant r),$$
$$\Sigma^{r+1} = A_1 - B_2, \qquad \Sigma^{r+k} = A_1 + B_k \quad (2 \leqslant k \leqslant s),$$

and also define $w = A_1 + B_1$. Writing $n = r + s$, we obtain a full rank subgroup $H = \langle \Sigma^1, \dots, \Sigma^n \rangle$ of $H_2(X; \mathbf{Z})$ such that $2H_2(X; \mathbf{Z})$ is contained in H. We also have

$$\Sigma^k \cdot \Sigma^k = 0, \qquad w \cdot \Sigma^k = 1.$$

Represent each Σ_k by a connected oriented surface of genus g_k. Then iterating the argument of Lemma 6.30, we obtain the following:

$$\mathbb{D}_{X,w}\big(\sum t_{2,k}\Sigma^k_{(2)} + \sum t_{3,k}\Sigma^k_{(3)}\big)$$

$$= e^{Q(\sum t_{2,k}\Sigma^k)/2 - Q(\sum t_{3,k}\Sigma^k)} \sum_{\substack{1 \leqslant k \leqslant n \\ (a_k,b_k) \in \mathcal{C}_{g_k}}} h^w_{a_1,b_1,\dots,a_n,b_n} e^{\sqrt{3}\sum a_k t_{2,k} + \sqrt{-3}\sum b_k t_{3,k}}$$

where each unindexed sum runs from $k = 1$ to $k = n$, and where $h^w_{a_1,b_1,\dots,a_n,b_n} \in \mathbf{C}$. Here $t_{j,k}$ are formal variables, for $j = 2, 3$ and $1 \leqslant k \leqslant n$. Now let $\Gamma, \Lambda \in H_2(X)$ be arbitrary. We may write $\Gamma = \sum x_k \Sigma^k$ and $\Lambda = \sum y_k \Sigma^k$ for some complex numbers x_k, y_k. Then specializing each $t_{2,k}$ to $x_k t_2$ and each $t_{3,k}$ to $y_k t_3$ gives the following expression:

$$\mathbb{D}_{X,w}(t_2\Gamma_{(2)} + t_3\Lambda_{(3)})$$

$$= e^{Q(t_2\Gamma)/2 - Q(t_3\Lambda)} \sum_{\substack{1 \leqslant k \leqslant n \\ (a_k,b_k) \in \mathcal{C}_{g_k}}} h^w_{a_1,b_1,\dots,a_n,b_n} e^{t_2\sqrt{3}\sum a_k x_k + t_3\sqrt{-3}\sum b_k y_k}$$

Write $\Sigma^k_\star \in H_2(X)$ for the dual basis of the Σ^k under the intersection pairing, so that $\Sigma^k_\star \cdot \Sigma^l = \delta_{kl}$. Note $2\Sigma^k_\star \in H_2(X; \mathbf{Z})$. Define

$$I := \{ i = (i_1, \dots, i_n) \in \mathbf{Z}^n \;\; | \;\; |i_k| < g_k \},$$

$$K_i := 2\sum_{k=1}^{n} i_k \Sigma^k_\star \in H_2(X; \mathbf{Z}) \quad \text{for} \quad i \in I. \tag{6.37}$$

Then we obtain (6.36) (setting $t_2 = t_3 = 1$) by letting i, j range over I and setting

$$c^w_{i,j} = h^w_{a_1,b_1,\dots,a_n,b_n}$$

where $i, j \in I$ are uniquely determined by $(a_1, b_1, \dots, a_n, b_n)$, and conversely, through the relations $i_k + j_k = a_k$ and $i_k - j_k = b_k$ for all $1 \leqslant k \leqslant n$.

Note that the argument of Lemma 6.30 shows that the classes K_i obtained above do not depend on w. Alternatively, without appealing to this point, one can take the union of the classes obtained for each w, where one ranges over one w for each class in $H^2(X; \mathbf{Z}/3)$, to eliminate any a priori dependency.

Next, we argue that the K_i are integral lifts of $w_2(X)$, which amounts to showing $K_i \cdot x \equiv x^2 \pmod 2$ for all $x \in H_2(X; \mathbf{Z})$. This is an adaptation of Step 3 in the proof of Theorem 2 from [Muñ00]. It is clear from the definition of K_i in (6.37) that $\Sigma^k \cdot K_i \equiv 0 \pmod 2$, which agrees mod 2 with $\Sigma^k \cdot \Sigma^k = 0$, and this verifies the claim on $H \subset H_2(X;\mathbf{Z})$. The general property used here in fact essentially follows from Lemma 6.30: if $\Sigma \subset X$ satisfies $\Sigma \cdot w \not\equiv 0$ and $\Sigma \cdot \Sigma = 0$, then $\Sigma \cdot K_i$ is even for any of the K_i.

Now suppose $x \in H_2(X;\mathbf{Z})\backslash H$. Then there is some k for which $x \cdot \Sigma^k \neq 0$. We can find $m \in \mathbf{Z}$ such that $x' := x + m\Sigma^k$ satisfies

$$N := (x')^2 \geqslant 0, \qquad w \cdot x' \not\equiv 0 \pmod 3.$$

(Here the property $w \cdot \Sigma^k = 1$ is used to obtain the second condition.) Now let $\widetilde{X}$ be X blown up at N points, and denote by $E_1, \dots, E_N$ the associated exceptional divisors. It follows from the blowup formulas of Theorem 6.18 that if $\{K_i\}$ are the classes in (6.36) for X, then classes associated to $\widetilde{X}$ are given by

$$K_i + \sum_{l=1}^{N} \varepsilon_l E_l, \qquad \varepsilon_l \in \{1, -1\}. \tag{6.38}$$

Consider $y = x' - E_1 - \cdots - E_N$. This satisfies $y^2 = 0$ and $y \cdot w \not\equiv 0 \pmod 3$. By the previous paragraph, we have that $y \cdot (K_i + \sum_{l=1}^{N} \varepsilon_l E_l)$ is even. On the other hand,

$$y \cdot (K_i + \sum_{l=1}^{N} \varepsilon_l E_l) \equiv x \cdot K_i + N \pmod 2.$$

Since $x^2 \equiv (x')^2 = N \pmod 2$, this proves the claim for x, and shows that each K_i is indeed characteristic.

We next consider the analogue of Step 4 in the proof of Theorem 2 from [Muñ00]. The goal is to show, upon setting $c_{i,j} = c^0_{i,j}$, that we have the relation

$$c^w_{i,j} = \zeta^{w \cdot \left(\frac{K_i - K_j}{2}\right)} c_{i,j}. \tag{6.39}$$

We now suppose $w^2 > 0$, as the invariants only depend on the mod 3 reduction of w, and every non-zero class in $H^2(X;\mathbf{Z})$ is mod 3 congruent to one with positive square. Consider again $\widetilde{X}$, the blowup of X at $N := w^2$ points, with exceptional divisors $E_1, \dots, E_N$. By the blowup formula, the classes associated to $\widetilde{X}$ are as in (6.38). Write

$$\widetilde{K}_i = K_i + \sum_{l=1}^{N} \varepsilon^i_l E_l, \qquad \widetilde{K}_j = K_j + \sum_{l=1}^{N} \varepsilon^j_l E_l$$

for two such classes. Then the blowup formula gives

$$c^{E_1}_{\widetilde{K}_i, \widetilde{K}_j} = \frac{q_1 p_2 \cdots p_N}{3^N 2} c_{K_i, K_j} \tag{6.40}$$

where the numbers p_k and q_k are defined as follows:

$$p_k = \begin{cases} 1/2, & \varepsilon^j_k = \varepsilon^i_k \\ 1, & \varepsilon^j_k \neq \varepsilon^i_k \end{cases} \qquad q_k = \begin{cases} 1/2, & \varepsilon^j_k = \varepsilon^i_k \\ \zeta^{-(\varepsilon^i_k - \varepsilon^j_k)/2}, & \varepsilon^j_k \neq \varepsilon^i_k \end{cases}$$

Consider $x = w - E_1 - \cdots - E_N$. Note that $x^2 = 0$ and $x \cdot E_1 \not\equiv 0 \pmod 3$. In this situation Lemma 6.30 provides the relationship

$$c^{w-E_2-\cdots-E_N}_{\widetilde{K}_i,\widetilde{K}_j} = \zeta^{\frac{1}{2}x\cdot\left(\widetilde{K}_i-\widetilde{K}_j\right)} c^{E_1}_{\widetilde{K}_i,\widetilde{K}_j} = \zeta^{\frac{1}{2}w\cdot(K_i-K_j)+\frac{1}{2}\sum_{l=2}^{N}(\varepsilon^i_l-\varepsilon^j_l)} c^{E_1}_{\widetilde{K}_i,\widetilde{K}_j}. \tag{6.41}$$

On the other hand, another application of the blowup formula yields

$$c^{w-E_2-\cdots-E_N}_{\widetilde{K}_i,\widetilde{K}_j} = \frac{p_1\overline{q}_2\cdots\overline{q}_N}{3^N} c^{w}_{K_i,K_j}. \tag{6.42}$$

Combining (6.40)–(6.42), we obtain the desired relation (6.39).

The claim that $c_{i,j} \in \mathbf{Q}[\zeta]$ follows from (6.36) and the fact that the invariants $D_{X,w}$ output rational values. These same observations also imply that $c_{i,j}$ is the complex conjugate of $c_{j,i}$. Furthermore, there is the general property

$$D_{X,-w}(z) = D_{X,w}(\tau(z)) \tag{6.43}$$

where $z \in \mathbf{A}^3(X)$ and $\tau : \mathbf{A}^3(X) \to \mathbf{A}^3(X)$ is the algebra homomorphism which maps $\alpha_{(r)}$ to $(-1)^r\alpha_{(r)}$; see [DX20, 2.10]. Taking $w = 0$, relations (6.43) and (6.36) yield $c_{i,j} = c_{j,i}$. We conclude that $c_{i,j}$ is real and hence $c_{i,j} \in \mathbf{Q}[\sqrt{3}]$.

Finally we consider the adjunction inequality (6.27). The proof of Step 5 in the proof of Theorem 2 from [Muñ00] carries over nearly verbatim. An argument in [KM95] reduces the proof to the case in which $N := \Sigma\cdot\Sigma > 0$. Consider again $\widetilde{X}$, the blowup of X at N points, and the proper transform $\widetilde{\Sigma} \subset \widetilde{X}$ of Σ, which represents the class $\Sigma - E_1 - \cdots - E_N$. As $\widetilde{\Sigma}\cdot\widetilde{\Sigma} = 0$ and $\widetilde{\Sigma}\cdot w \not\equiv 0 \pmod 3$ for $w = E_1$, Lemma 6.30 yields

$$2g - 2 \geqslant \left| \left(K_i + \sum \varepsilon_l E_l\right) \cdot (\Sigma - E_1 - \cdots - E_N) \right|$$

for all of the associated classes K_i of X, and all $\varepsilon \in \{1,-1\}^N$. This implies the desired inequality (6.27), and completes the proof of the theorem. □

7 A non-vanishing theorem for symplectic 4-manifolds

In this section, we prove Theorem 1.14 of the introduction, which we restate here:

Theorem 7.1. *Let X be a closed symplectic 4-manifold with $b^+(X) > 1$. Then the invariant $D^3_{X,w}$ is non-trivial for all $w \in H^2(X;\mathbf{Z})$.*

As a consequence of Theorem 7.1, we have the following non-vanishing result for admissible bundles. In the same way that we deduce Theorem 1.3 from Corollary 5.21, the following non-vanishing result can be used to give another proof of Theorem 1.3.

Corollary 7.2. *Let (Y,γ) be an admissible pair such that Y is irreducible. Then the instanton Floer homology group $I^3_*(Y,\gamma)$ is non-trivial.*

Proof. The corollary is a consequence of Theorem 7.1 and a result about embeddings of 3-manifolds into symplectic manifolds. Since (Y, γ) is admissible, we have $b_1(Y) > 0$. In particular, Y can be embedded in a symplectic manifold X as a separating submanifold such that the map $H^2(X; \mathbf{Z}) \to H^2(Y; \mathbf{Z})$ is surjective and the two components X_1 and X_2 obtained by cutting X along Y have $b^+ > 0$. This follows from Gabai's theorem about the existence of taut foliations on 3-manifolds with $b_1 > 0$ [Gab83] and [KM04, Proposition 15]. The latter is obtained by combining various earlier results [Eli04, Etn04, ET98, KM04] (see also [KM07, Section 41.3]). Our control on $H^2(X; \mathbf{Z})$ implies that there is a 2-cycle w on X whose intersection with Y is homologous to γ. Using Theorem 7.1, we know

$$\mathbb{D}^3_{X,w}(\Gamma_{(2)} + \Lambda_{(3)})$$

is non-trivial for some $\Gamma, \Lambda \in H_2(X; \mathbf{Z})$ where Γ and Λ are represented by embedded surfaces whose intersection with Y are respectively equal to c and l. Now we can use the pairing formula (6.5) to see that the Fukaya–Floer homology group $\mathbb{I}^3_*(Y, \gamma, L)$ with L given by the homology classes of c and l is non-trivial. The non-vanishing of this Fukaya–Floer homology group implies that $\mathbb{I}^{3,j}_*(Y, \gamma, L)$ is non-zero for some j. The spectral sequence from $I_*(Y, \gamma) \otimes \mathbf{C}^{2^{2j}}$ to $\mathbb{I}^{3,j}_*(Y, \gamma, L)$ implies that $I^3_*(Y, \gamma)$ is non-zero. □

Proof of Theorem 7.1. After possibly perturbing the symplectic form of X and then rescaling, we can assume that the symplectic form ω of X represents an integral cohomology class. Now [Don99, Theorem 2] implies that X admits a (topological) Lefschetz pencil such that the fibers are symplectic subvarieties representing the Ponicaré dual of $k[\omega]$ where k is a large enough integer. In particular, the base locus of this Lefschetz pencil is given by a non-empty set of points $\{x_1, \cdots, x_m\}$, and by blowing up X at these points, we obtain $\widehat{X}$, which is a Lefschetz fibration over S^2 where a generic fiber F (obtained as the proper transform of a fiber of the Lefschetz pencil) represents the cohomology class

$$k[\omega] - E_1 - \cdots - E_m$$

with E_i the exceptional classes. Taking k large enough, we may also assume that the genus of F is as large as we wish and all fibers of the Lefschetz fibration are irreducible. The latter claim is [Smi01, Theorem 3.10] and the former follows from adjunction formula. (See, for example, (3.9) in [Smi01].) The 2-cycle w in X induces a cycle in $\widehat{X}$ and if necessary we add $\pm E_1$ to this cycle to guarantee that the resulting cycle $\widehat{w}$ in $\widehat{X}$ satisfies $\widehat{w} \cdot F \equiv 1 \bmod 3$. By Theorem 6.18, if we show that $D^3_{\widehat{X},\widehat{w}}$ is non-trivial, then $D^3_{X,w}$ is also non-trivial.

We decompose the base S^2 of the Lefschetz fibration structure on $\widehat{X}$ as a union $D_- \cup A_1 \cup \cdots A_l \cup D_+$ such that $D_\pm$ are discs and the A_i are annuli, and the Lefschetz fibration has no critical point over the discs and exactly one critical point over each annulus. This induces a decomposition of $\widehat{X}$ as follows:

$$\widehat{X} = D_- \times F \cup W_1 \cup \cdots \cup W_l \cup D_+ \times F$$

where $W_i : Y_{i-1} \to Y_i$ is a cobordism that admits a Lefschetz fibration over the annulus A_i with one singular fiber. In particular, Y_i fibers over S^1 with fiber F. We regard $D_- \times F$

as a cobordism from the empty set to $Y_0 = S^1 \times F$ and $D_+ \times F$ as a cobordism from $Y_l = S^1 \times F$ to the empty set. Without loss of generality, we can assume that the intersection of $\widehat{w}$ and Y_i is transversal and we write γ_i for the induced 1-cycle on Y_i. We also denote the intersection of $\widehat{w}$ with $D_\pm \times F$ and W_i by $w_\pm$ and w_i. Our assumption on $\widehat{w} \cdot F$ implies that (Y_i, γ_i) is 3-admissible. (In the special case of γ_0 and γ_l, they are given by circle fibers of Y_0 and Y_l.) The 3-manifolds Y_i can be regarded as a closure of the product sutured manifold. In particular, we can use Theorem 1.5 to see that $I^3_*(Y_i, \gamma_i|F) = \mathbf{C}$.

As the Floer homology of $(S^1 \times F, \gamma_1)$ is generated by relative invariants of $D^2 \times F$, there exist polynomials $p_\pm \in \mathbf{C}[x, y, z]$ such that

$$D^3_{D_- \times F, w_-}(p_-(x_{(2)}, F_{(2)}, F_{(3)})) \in I^3_*(Y_0, \gamma_0|F), \tag{7.3}$$

$$D^3_{D_+ \times F, w_+}(p_+(x_{(2)}, F_{(2)}, F_{(3)})) : I^3_*(Y_l, \gamma_l|F) \to \mathbf{C} \tag{7.4}$$

are non-trivial. The gluing formula expresses the invariant

$$D^3_{\widehat{X}, \widehat{w}}(p_+ p_-(x_{(2)}, F_{(2)}, F_{(3)}))$$

in terms of a composition of the two quantities (7.3), (7.4) and the maps

$$I^3_*(W_l, w_l) : I^3_*(Y_{l-1}, \gamma_{l-1}|F) \to I^3_*(Y_l, \gamma_l|F). \tag{7.5}$$

Thus to prove our claim, it suffices to show that (7.5) is non-zero.

The cobordism $W_l : Y_{l-1} \to Y_l$ can be decomposed further as the composition of the following two 4-dimensional cobordisms:

$$\mathcal{L}_l : \varnothing \to Y_{\phi_l}, \qquad \mathcal{P}_l : Y_{l-1} \sqcup Y_{\phi_l} \to Y_l,$$

where Y_{δ_l} is the mapping torus of a positive Dehn twist along a non-separating simple closed curve in F. Here we are using the fact that the fibers of our Lefschetz fibration are irreducible. The positive Dehn twist δ_l is determined by the property that if Y_i is the mapping torus of the diffeomorphism $\phi_i : F \to F$, then $\phi_l = \delta_l \circ \phi_{l-1}$. The cycle w_l induces the cycles c_l and c'_l on $\mathcal{L}_l$ and $\mathcal{P}_l$. We also write ε_l for the induced cycle on Y_{δ_l}. The excision theorem of [DX20] implies that the cobordism map

$$I^3_*(\mathcal{P}_l, c'_l) : I^3_*(Y_{l-1}, \gamma_{l-1}|F) \otimes I^3_*(Y_{\delta_l}, \varepsilon_l|F) \to I^3_*(Y_l, \gamma_l|F) \tag{7.6}$$

is an isomorphism of 1-dimensional vector spaces. The following lemma and the non-vanishing of (7.6) implies the non-vanishing of (7.5), completing the proof. □

Lemma 7.7. *For an oriented closed surface F, let $\mathcal{L}$ be a Lefschetz fibration over the 2-dimensional disc with one irreducible singular fiber. Let c be a 2-cycle on $\mathcal{L}$ such that $c \cdot F \equiv 1$ mod 3. Then $I^3_*(\mathcal{L}, c)$ has a non-trivial component in $I^3_*(Y, \gamma|F)$ where (Y, γ) is the boundary of $(\mathcal{L}, c)$.*

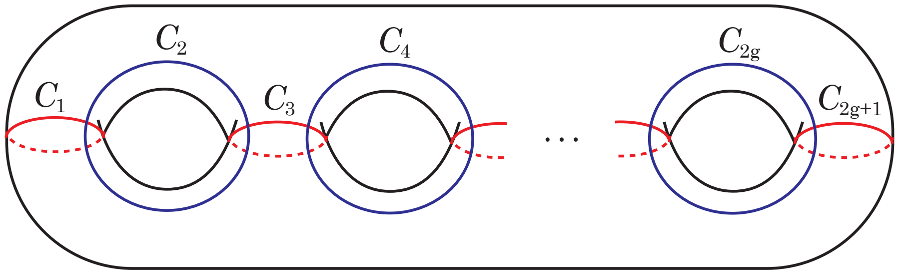

Figure 1: The curves C_i for $1 \leqslant i \leqslant 2g+1$ on a surface of genus g.

Proof. The 4-manifold $\mathcal{L}$ can be embedded into any closed 4-manifold X together with the structure of a genus $g = g(F)$ Lefschetz fibration, which has at least one irreducible singular fiber. For instance, we can take the elliptic surface $X = E(g+1)$, which has a genus g Lefschetz fibration in addition to its standard elliptic fibration. In fact, we may take a Lefschetz fibration with the singular fibers given by the monodromies

$$(\psi_1, \psi_2, \cdots, \psi_{2g-1}, \psi_{2g-1}, \cdots, \psi_2, \psi_1)^4$$

where ψ_i denotes the Dehn twist along the simple closed curve C_i in Figure 1. See [GS99, Chapter 8]. We still denote the fibers of this Lefschetz fibration by F. Then the algebraic intersection of F and a fiber f of the elliptic fibration of $E(g+1)$ is equal to 2.

The 4-manifold $\mathcal{L}$ can be identified with a regular neighborhood of the singular fiber corresponding to ψ_1. The complement of $\mathcal{L}$ determines another Lefschetz fibration $\mathcal{Z}$ over a disc. This manifold has the homotopy type of F where we glue 2-cells to it along simple close curves corresponding to the Dehn twists involved in the Lefschetz fibration structure of $\mathcal{Z}$. In particular, $H_1(\mathcal{Z})$ is trivial, and hence there is a 2-cycle c' on $\mathcal{Z}$ whose restriction to Y is γ. The 2-cycles c and c' can be glued to each other to form a 2-cycle $\tilde{c}$ on $E(g+1)$. We may further split $(\mathcal{Z}, c)$ as the composition of (Z_0, c_0) and (Z_1, c_1) such that Z_1 is a regular neighborhood of regular fiber of the Lefschetz fibration of $\mathcal{Z}$ and Z_1 is the complement. In particular, Z_1 is diffeomorphic to $D^2 \times F$ and we can assume that $c_1 = D^2 \times \{x\}$ for $x \in F$. The pair (Z_0, c_0) can be regarded as a cobordism from (Y, γ) to $(S^1 \times F, \gamma_1)$. To prove the claim, it suffices to show that there is a polynomial $Q \in \mathbf{C}[x, y]$ such that

$$\widehat{D}^3_{Z_0,c_0}(Q(F_{(2)}, F_{(3)})) \circ I^3_*(\mathcal{L}, c) \tag{7.8}$$

is a non-zero element of $I^3_*(S^1 \times F, \gamma_1|F)$. Then functoriality implies that $I^3_*(\mathcal{L}, c)$ has a non-trivial component in $I^3_*(Y, \gamma|F)$.

The polynomial p can be constructed as in [DX20, Proposition 5.7] using the calculation of $U(3)$ invariants of elliptic surfaces in [DX20]. For the pair $(E(g+1), w)$, we have

$$\mathbb{D}^3_{E(g+1),w}(t_2F_{(2)} + t_3F_{(3)}) = \left(\frac{2}{3}\cosh(2\sqrt{3}t_2) - \frac{2}{3}\cosh(-\frac{2\pi i}{3}w \cdot f + 2\sqrt{3}it_3)\right)^{g-1}.$$

In particular, it is a power series of the form

$$\mathbb{D}^3_{E(g+1),w}(t_2F_{(2)}+t_3F_{(3)})=\sum_{a,b} d^w_{a,b}e^{2\sqrt{3}at_2+2\sqrt{3}ibt_3} \tag{7.9}$$

where $d^w_{a,b}$ are constant coefficients, $a+b$ has the same parity as $g-1$ and $|a|+|b|\leqslant g-1$. For instance, we have $d^w_{g-1,0}=(2/3)^{g-1}$. We may use the above identities to compute $\widehat{D}_{X,w}(P(F_{(2)},F_{(3)}))$ for any polynomial P. In fact, using the notation in (7.9) we have

$$\widehat{D}^3_{E(g+1),w}(P(F_{(2)},F_{(3)}))=\sum_{a,b} d^w_{a,b}P(a,b). \tag{7.10}$$

Now let Q be a polynomial such that $Q(g-1,0)=1$, and $Q(a,b)=0$ for any other (a,b) as above. Using (7.11) and the fact that $E(n)$ with $n\geqslant 2$ is $U(3)$ simple type, we have

$$\widehat{D}^3_{E(g+1),w}(R(x_{(2)},x_{(3)},F_{(2)},F_{(3)})Q(F_{(2)},F_{(3)}))=\left(\frac{2}{3}\right)^{g-1}R(3,0,g-1,0) \tag{7.11}$$

for any polynomial $R\in\mathbf{C}[v,w,x,y]$. We claim that this polynomial Q satisfies the required property for (7.8).

Gluing (Z_0,c_0) and $(\mathcal{L},c)$ produces a pair $(\widetilde{Z}_0,\widetilde{c}_0)$ with boundary $(S^1\times F,\gamma_1)$, and by functoriality (7.8) is equal to the following:

$$\widehat{D}^3_{\widetilde{Z}_0,\widetilde{c}_0}(Q(F_{(2)},F_{(3)})). \tag{7.12}$$

Using the functoriality of instanton Floer homology again, we see that

$$\begin{aligned}\langle\widehat{D}^3_{\widetilde{Z}_0,\widetilde{c}_0}(Q(F_{(2)},F_{(3)})),\ D^3_{Z_1,c_1+l\cdot F}(R(x_{(2)},x_{(3)},F_{(2)},F_{(3)}))\rangle&\\ =\widehat{D}^3_{E(g+1),\widetilde{c}+l\cdot F}(R(x_{(2)},x_{(3)},F_{(2)},F_{(3)})Q(F_{(2)},F_{(3)})).&\end{aligned}$$

Since instanton Floer homology of $(S^1\times F,\gamma_1)$ is generated by elements of the form $D^3_{Z_1,c_1+l\cdot F}(R(x_{(2)},x_{(3)},F_{(2)},F_{(3)}))$, we may combine the above pairing formula and (7.11) to see that (7.12) is non-zero and belongs to the eigenspace $I^3_*(S^1\times F,\gamma_1|F)$. □

8 $U(N)$ framed instanton homology

In this section we study the $U(N)$ framed instanton homology for 3-manifolds. These groups were essentially introduced by Kronheimer and Mrowka [KM11b], and have been extensively studied in the $N=2$ case, see for example [KM11a, Sca15, BS23]. After establishing some basic properties of $U(N)$ framed instanton homology, we compute its Euler characteristic and state a connected sum theorem. In the final subsection we discuss a decomposition result for cobordism maps in the $N=3$ case, which follows from an adaptation of the $U(3)$ Structure Theorem in this setting.

8.1 Definition

Let Y be a closed, oriented, connected 3-manifold. Delete a small embedded open 3-ball from Y to obtain M, which has a 2-sphere boundary. Let α be any simple closed curve on the boundary of M. Define the $U(3)$ *framed instanton homology of* Y as follows:

$$I_*^{\#,3}(Y) = SHI_*^3(M, \alpha).$$

More concretely, the $U(3)$ framed instanton homology is given as

$$I_*^{\#,3}(Y) = I_*^3(Y\#T^3, \gamma|R)$$

where γ is the 1-cycle $S^1 \times \{x\}$ in $T^3 = S^1 \times T^2$ and $R = \{y\} \times T^2$, where $x \in T^2$ and $y \in S^1$. As $\mu_2(R) = \mu_3(R) = \beta_3 = 0$ on the group $I_*^3(Y\#T^3)$, it follows that $I_*^{\#,3}(Y)$ is defined as the (3)-eigenspace of β_2 acting on $I_*^3(Y\#T^3)$. Note that we have already encountered these groups at the end of Section 5.

More generally, we define the $U(N)$ framed instanton homology for $N \geqslant 2$ as

$$I_*^{\#,N}(Y) = I_*^N(Y\#T^3, \gamma|R) \tag{8.1}$$

where the notation on the right-side denotes the (N)-eigenspace of the operator β_2 acting on $I_*^N(Y\#T^3, \gamma)$. This construction is due to Kronheimer and Mrowka, see [KM11b, §4.1]. The $N = 2$ version has been studied in various settings (see e.g. [Sca15]), often motivated by Kronheimer and Mrowka's conjecture [KM10b, §7.9] that $I^{\#,2}(Y)$ is isomorphic to Ozsváth and Szabó's Heegaard Floer group $\widehat{HF}(Y)$ with complex coefficients.

Remark 8.2. Note that definition (8.1) allows one to use any coefficient ring when defining $U(N)$ framed instanton homology. In what follows, we will continue to assume that complex coefficients are used.

As $I_*^N(Y\#T^3, \gamma)$ is relatively $\mathbf{Z}/4N$-graded and β_2 has degree 4, the group (8.1) inherits a relative $\mathbf{Z}/4$-grading. This can be lifted to an absolute $\mathbf{Z}/4$-grading, just as in the $N = 2$ case; the discussion in [Sca15, §7.3] adapts in a straightforward manner.

For our purposes, we only need to specify an absolute $\mathbf{Z}/2$-grading on $I^{\#,N}(Y)$. To do this, it suffices to define an absolute $\mathbf{Z}/2$-grading on $I^N(Y, \omega)$ for any N-admissible pair (Y, ω). For a critical point ρ which is a generator of the complex defining $I_*^N(Y, \omega)$, set

$$\mathrm{gr}(\rho) = \mathrm{ind}(A) + (N^2 - 1)(b_1(X) - b^+(X) + b_1(Y) - 1) \pmod 2 \tag{8.3}$$

where $\mathrm{ind}(A)$ is the index of D_A, the (perturbed) ASD operator associated to a $PU(N)$-connection A over a 4-manifold X with cylindrical end $Y \times [0, \infty)$, with A restricting to the pullback of ρ over $Y \times [0, \infty)$. This is well-defined by an argument analogous to the one given in [Don02, §5.6], using the index formulas found in [Kro05], for example.

Note from the construction of framed instanton homology that

$$\dim I^{\#,N}(Y) = \frac{1}{N} \dim I_*^N(Y\#T^3, \gamma).$$

Consider $Y = S^3$. As $I_*^N(T^3, \gamma)$ is of dimension N, generated by N non-degenerate flat connections (see [Kro05, KM11b]), the dimension of $I^{\#,N}(S^3)$ is 1.

Note that the (3)-eigenspace of β_2 acting on $I_*(Y \# T^3)$ agrees with the (1)-eigenspace of the operator ε. The action of ε can also be viewed as the action of a certain $PU(N)$-gauge transformation supported on the T^3-factor. From this viewpoint, which will be adapted below, $I^{\#}(Y, \gamma)$ is the Morse homology of a (perturbed) Chern–Simons functional on a configuration space of $PU(N)$-connections which is quotiented by a slightly larger gauge group, and its critical set (in the unpertubed case) is homeomorphic to

$$R^N(Y) := \mathrm{Hom}(\pi_1(Y), SU(N)),$$

the $SU(N)$ representation space of $\pi_1(Y)$. Note that the quotient of $R^N(Y)$ by the action of conjugation, denoted $\mathfrak{X}^N(Y)$, is the $SU(N)$ character variety of Y.

8.2 Euler characteristic

In [Sca15] it was shown that $I^{\#,2}(Y)$ has Euler characteristic equal to $|H_1(Y; \mathbf{Z})|$ if $b_1(Y) = 0$, and is otherwise zero. This fact generalizes as follows.

Theorem 8.4. *For any $N \geqslant 2$ and any closed, oriented, connected 3-manifold Y:*

$$\chi\left(I^{\#,N}(Y)\right) = \begin{cases} |H_1(Y; \mathbf{Z})|^{N-1}, & b_1(Y) = 0 \\ 0, & b_1(Y) > 0 \end{cases} \tag{8.5}$$

Proof. We first explain the proof under the assumption that $b_1(Y) = 0$, $\mathfrak{X}^N(Y)$ is a finite set of non-degenerate points. In particular, $R^N(Y)$ is Morse–Bott nondegenerate for the Chern–Simons functional on $(Y \# T^3, \gamma)$. In particular, we have a homeomorphism of spaces

$$R^N(Y) \cong \bigsqcup_{[\rho] \in \mathfrak{X}^N(Y)} SU(N)/\Gamma_\rho \tag{8.6}$$

where $\Gamma_\rho \subset SU(N)$ denotes the stabilizer of ρ under the conjugation action. The stabilizer Γ_ρ is isomorphic to a group of the form

$$S(U(n_1) \times U(n_2) \times \cdots \times U(n_k))$$

where $\sum n_i \leqslant N$. Let us say that ρ is *abelian* if $\sum n_i = N$. This terminology is justified by the fact that ρ is abelian if and only if it factors through the abelianization $H_1(Y; \mathbf{Z})$; an equivalent condition is that the stabilizer Γ_ρ has the same rank as $SU(N)$. (Recall that the rank of a compact Lie group is the dimension of a maximal torus.)

A abelian $SU(N)$ representation ρ may be constructed by taking $N-1$ homomorphisms $\rho_i : H_1(Y; \mathbf{Z}) \to U(1)$ for $i = 1, \ldots, N-1$ and composing

$$\rho_1 \oplus \cdots \oplus \rho_{N-1} \oplus (\rho_1 \cdots \rho_{N-1})^{-1} : H_1(Y; \mathbf{Z}) \to SU(N)$$

with the natural surjection $\pi_1(Y) \to H_1(Y;\mathbf{Z})$. This constructs $|H_1(Y;\mathbf{Z})|^{N-1}$ abelian representations; call these *standard*. Every abelian representation is conjugate to a standard one, but some standard abelian representations are equivalent by conjugation. Conjugation induces an action of the Weyl group of $SU(N)$, the symmetric group S_N, on the set of standard abelian representations. The orbit-stabilizer formula for this S_N-action yields

$$|H_1(Y;\mathbf{Z})|^{N-1} = \sum_{[\rho]\in\mathfrak{R}^N(Y)} |W_{SU(N)}|/|W_{\Gamma_\rho}| \tag{8.7}$$

where $\mathfrak{R}^N(Y) \subset \mathfrak{X}^N(Y)$ is the subset of abelian classes. The notation W_G denotes the Weyl group of G. In writing this formula we have identified the stabilizer of ρ under the $S_N = W_{SU(N)}$ action with the Weyl group of Γ_ρ. A result of Hopf and Samelson [HS41] says that a connected homogeneous space G/H, where G is a compact Lie group and H is a closed subgroup, has Euler characteristic

$$\chi(G/H) = \begin{cases} |W_G|/|W_H|, & \text{rank}(G) = \text{rank}(H) \\ 0, & \text{rank}(G) > \text{rank}(H) \end{cases} \tag{8.8}$$

Combining (8.8), (8.6), (8.7), and the earlier observation that ρ is abelian if and only if the rank of Γ_ρ is that of $SU(N)$, we obtain

$$\chi(R^N(Y)) = |H_1(Y;\mathbf{Z})|^{N-1}.$$

A small perturbation used in defining $I_*^{\#,N}(Y\#T^3,\gamma)$ can be chosen so that the orbit of ρ appearing in (8.6) is replaced by the set of critical points $\{\alpha_i\}$ of a Morse function on $SU(N)/\Gamma_\rho$. The mod 2 grading of α_i is given by its Morse index plus the mod 2 grading of ρ as defined by equation (8.3). Thus the relation

$$\chi\left(I^{\#,N}(Y)\right) = \chi(R^N(Y)) = |H_1(Y;\mathbf{Z})|^{N-1} \tag{8.9}$$

will hold in the case at hand once it is shown that the mod 2 grading of each abelian critical point is even. In what follows, we represent $\rho \in R^N(Y)$ by a connection $\alpha\#\beta$ on $Y\#T^3$, where α is a flat $SU(N)$ connection on Y and β is one of the N flat non-degenerate $PU(N)$ connections on T^3 compatible with the bundle data γ.

Let X be a 4-manifold with boundary Y. Denote by W the cobordism from Y to $Y\#T^3$ which is topologically the boundary sum of $Y \times I$ with $D^2 \times T^2$. Write X' for the union of X and W along Y. Let A_X be a $PU(N)$ connection on X with a cylindrical end attached, restricting to the pullback of α over the end, and $\text{ind}^-(A_X)$ the index of the associated ASD operator with exponential decay (see for example [Don02, §3.3.1]). Let A_W be a $PU(N)$ connection on W with cylindrical ends attached, equal to the pullback of α on the incoming end and that of $\alpha\#\beta$ on the outgoing end. Then by index additivity we have

$$\begin{aligned} \text{gr}(\alpha\#\beta) &\equiv \text{ind}(A) + (N^2-1)(b_1(X') - b^+(X') + b_1(Y\#T^3) - 1) \qquad (8.10)\\ &\equiv \text{ind}^-(A_X) + \text{ind}^{+-}(A_W) + (N^2-1)(b_1(X) - b^+(X)) \pmod 2 \end{aligned}$$

Here $\text{ind}^{+-}(A_W)$ is the index of the ASD operator associated to A_W with exponential growth at the incoming end and exponential decay at the outgoing end; see [DX20, §2.2] for this setup. By the Atiyah–Patodi–Singer theorem [APS75a] (see also [DX20, Eq. 2.16]),

$$\begin{aligned}\text{ind}^{+-}(A_W) &= 4N\kappa(A_W) - \frac{N^2-1}{2}(\chi(W)+\sigma(W)) \\ &\quad + \frac{1}{2}\left(h^0(\alpha)+h^1(\alpha)-h^0(\alpha\#\beta)-h^1(\alpha\#\beta)-\rho_{\text{ad}\alpha}(Y)+\rho_{\text{ad}(\alpha\#\beta)}(Y\#T^3)\right).\end{aligned}$$

Here $h^i(\alpha)$ is the dimension of $H^i(Y;\text{ad}\alpha)$, and so forth. By our current assumptions, we have $h^0(\alpha\#\beta) = h^1(\alpha) = 0$ and $h^0(\alpha) = \dim\Gamma_\rho$, while $h^1(\alpha\#\beta) = \dim SU(N)/\Gamma_\rho$. We also choose A_W to be a flat connection, obtained by gluing a flat connection extending β over $D^2\times T^2$ to the product flat connection induced by α on $Y\times I$ and extending by translation to cylindrical ends. (That β extends to a flat connection over $D^2\times T^2$ is easily verified by the description in [KM11b, §4.1].) By $\kappa(A_W)=0$, $\chi(W)=-1$, $\sigma(W)=0$:

$$\text{ind}^{+-}(A_W) = \dim\Gamma_\rho - \frac{1}{2}(\rho_{\text{ad}\alpha}(Y)-\rho_{\text{ad}(\alpha\#\beta)}(Y\#T^3)).$$

On the other hand, by Atiyah–Patodi–Singer's result [APS75b, Thm. 2.4], we have

$$\rho_{\text{ad}\alpha}(Y)-\rho_{\text{ad}(\alpha\#\beta)}(Y\#T^3) = (N^2-1)\sigma(W) - \sigma_{\text{ad}A_W}(W) \tag{8.11}$$

Here we use that the adjoint bundle has rank N^2-1, see (8.13). A computation using the Mayer–Vietoris sequence with local coefficients shows $H^2(W;\text{ad}A_W)=0$ and hence the right side of (8.11) vanishes. Thus $\text{ind}^{+-}(A_W)=\dim\Gamma_\rho$. Plugging into (8.10) yields

$$\text{gr}(\alpha\#\beta) \equiv \text{ind}^-(A_X) + \dim\Gamma_\rho + (N^2-1)(b_1(X)-b^+(X)) \pmod 2 \tag{8.12}$$

Now suppose ρ is abelian. Then α is compatible with a splitting $L_1\oplus\cdots\oplus L_N$ where each L_i is a complex line bundle. The associated adjoint bundle is isomorphic to

$$\bigoplus_{i<j} L_i\otimes L_j^{-1}\oplus \underline{\mathbf{R}}^{N-1} \tag{8.13}$$

We may choose X such that $H^2(X;\mathbf{Z})\to H^2(Y;\mathbf{Z})$ is a surjection; then we may choose line bundles $\widetilde{L}_i$ over X which extend the L_i. Further, choose A_X so that $\text{ad}A_X$ splits as $\oplus_{i<j}A_{ij}\oplus\Theta$ where Θ is a trivial connection on $\underline{\mathbf{R}}^{N-1}$ and A_{ij} is a $U(1)$ connection on $\widetilde{L}_i\otimes\widetilde{L}_j^{-1}$. With these choices, we compute

$$\text{ind}^-(A_X) \equiv -(N^2-1)(1-b_1(X)+b^+(X)) \pmod 2 \tag{8.14}$$

Indeed, the index of A_X splits into a sum; the indices associated to the A_{ij} are even, because the relevant operators are complex linear, and the index associated to Θ is (N^2-1) times the index of the standard ASD operator on X. We then obtain from (8.12):

$$\text{gr}(\alpha\#\beta) \equiv \dim\Gamma_\rho - (N^2-1) \equiv 0 \pmod 2$$

Here we have used that $\dim \Gamma_\rho = \sum n_i^2 - 1$ for some non-negative integers n_i which satisfy $\sum n_i = N$. This completes the proof of claim (8.9) under the given assumptions.

In the general case for $b_1(Y) = 0$, a holonomy perturbation must be used. When $N = 2$, it is explained in [Eis23, Theorem 3.6] that there are small holonomy perturbations for Y such that the critical set of the Chern–Simons functional is discrete and non-degenerate, and the corresponding orbits on the framed configuration space are Morse–Bott non-degenerate. In our case, we use such a perturbation on $Y \# T^3$ which is supported on Y. For the above argument to adapt, it is important that for a small enough such perturbation, the number of abelian critical points and their stabilizer-types remain the same; this is true because these reducibles are cut out transversely within the subspace of the configuration space consisting of abelian connections.

When $b_1(Y) > 0$, the abelian representations in $R^N(Y)$ form a disjoint union of tori, each of dimension $b_1(Y)$. In the simplified version of the above argument, these tori now contribute zero to the Euler characteristic. Adapting the above argument, with similar remarks regarding perturbations, gives $\chi(I^{\#,N}(Y)) = 0$ in this case. □

Remark 8.15. The absolute $\mathbf{Z}/2$-grading used here agrees with that of [CDX17, Proposition 6.20]. In that reference, the $\mathbf{Z}/2$-grading is determined by the conditions that (i) for any cobordism $(W, c) : (Y, \omega) \to (Y', \omega')$ between N-admissible pairs, the degree of the corresponding cobordism map is the parity of

$$\frac{N^2 - 1}{2}(\chi(W) + \sigma(W) + b_0(Y') - b_0(Y) + b_1(Y') - b_1(Y));$$

and (ii) the generator of $I^N_*(\varnothing)$, which is by convention 1-dimensional, is supported in even degree. Condition (i) follows from the definition (8.3) along the same lines as [Sca15, Prop. 7.1]. Furthermore, the normalization condition (ii) is equivalent, assuming (i), to the condition that $I^{\#,N}(S^3)$ is supported in even degree.

The framed instanton homology can also be defined for any 3-manifold Y with a 1-cycle $\omega \subset Y$. In this case the $U(N)$ framed instanton homology is denoted

$$I^{\#,N}_*(Y, \omega) \tag{8.16}$$

and is defined as the (N)-eigenspace of the operator β_2 acting on $I^N_*(Y \# T^3, \omega \cup \gamma)$. The isomorphism type of this group only depends on Y and $[\omega] \in H_1(Y; \mathbf{Z}/N)$, and sometimes we conflate ω with its homology class, or its Poincaré dual. The argument of Theorem 8.4 can be adapted to show that the Euler characteristic of (8.16) is also given by the right side of (8.5), and is in particular independent of ω.

Remark 8.17. Theorem 8.4 (and its extension to (8.16) mentioned above) is compatible with the surgery exact $(N + 1)$-gons of [CDX17], which were proven for $N \leqslant 4$. In particular, it satisfies the Euler characteristic relation [CDX17, Cor. 1.9], giving evidence for the existence of surgery exact $(N + 1)$-gons for $N > 4$.

It follows from a result of Borel [Bor53] that if G is a compact connected Lie group and H is a closed connected subgroup with rank equal to that of G, then the cohomology of G/H with complex coefficients is supported in even degrees. In particular, in the case that $b_1(Y) = 0$, if $R^N(Y)$ consists entirely of abelian representations and is Morse–Bott non-degenerate for the Chern–Simons functional, then

$$\dim I^{\#,N}(Y) = |H_1(Y;\mathbf{Z})|^{N-1}. \tag{8.18}$$

This occurs in the case that Y is a lens space. The condition (8.18), that the dimension of the $U(N)$ framed instanton homology is equal to its Euler characteristic, is a natural generalization of the $U(2)$ instanton L-space condition [BS23]. A natural question is whether the class of 3-manifolds defined by the condition (8.18) depends on N. In the case that Y is an $U(2)$ instanton L-space and also satisfies (8.18) for some $N > 2$, we have

$$I^{\#,N}(Y) \cong I^{\#,2}(Y)^{\otimes(N-1)}. \tag{8.19}$$

Thus we are led to ask: is there an example of a 3-manifold Y for which (8.19) does not hold for some $N > 2$?

Remark 8.20. In the above discussion, one may also include the case $N = 1$. The $U(1)$ framed instanton homology of Y is a special case of the plane Floer homology of the first author [Dae], and is isomorphic to the homology of the Jacobian torus of Y.

8.3 A product formula for connected sums

The $U(N)$ framed instanton homology behaves in a simple way with respect to connected sums. The following generalizes a known result in the $N = 2$ case.

Theorem 8.21. *Let (Y,ω) and (Y',ω') be connected 3-manifolds with 1-cycles. Then*

$$I^{\#,N}(Y\#Y',\omega\cup\omega') \cong I^{\#,N}(Y,\omega)\otimes I^{\#,N}(Y',\omega') \tag{8.22}$$

The proof of this result is analogous to the proof in the $N = 2$ case, which is a slight variation of the argument given in [KM11a, Cor. 5.9]. The main device used is genus 1 excision, which holds for all N, just as in the $N = 2$ case, by the simple description of the Floer homology of (T^3,γ) as given in [Kro05, KM11b]. With genus 1 excision, the key observation in proving (8.22) is that one may cut open $Y\#T^3$ and $Y'\#T^3$ along the copies of 2-tori labelled R in each 3-torus, and reglue the resulting boundary components so as to form a connected 3-manifold diffeomorphic to $Y\#Y'\#T^3$. From the proof it is also easy to see that, just as in the $N = 2$ case, the isomorphism (8.22) preserves $\mathbf{Z}/2$-gradings, and is natural with respect to "split" cobordisms.

8.4 A decomposition result for cobordism maps

Despite the various properties of $U(N)$ framed instanton homology discussed above, for $N > 2$ very few computations of these groups are currently known. In the $N = 2$ case, many computations have been aided by the use of Floer's exact triangle [Flo95, Sca15]. The surgery exact $(N+1)$-gons of [CDX17], proved in the cases $N \leqslant 4$, provide a generalization that may be useful for computations in the higher rank cases. Another tool that has been very useful in the $N = 2$ case, as is illustrated in the work of Baldwin and Sivek [BS23], is a decomposition result for cobordism maps. In this subsection we explain how to obtain an analogous decomposition result in the $N = 3$ case. This is essentially an adaptation of Theorem 6.26 to the setting of cobordism maps in $U(3)$ framed instanton homology.

Let $(X, w) : (Y, \omega) \to (Y', \omega')$ be a cobordism of pairs, where w is a 2-cycle restricting to ω and ω' at the boundary components. Here and below we suppose X, Y and Y' are connected. There is an associated cobordism map of framed instanton homology groups

$$I^{\#,N}(X, w) : I^{\#,N}(Y, \omega) \to I^{\#,N}(Y', \omega')$$

obtained by choosing a path c embedded in $X \backslash w$ and splicing $I \times T^3$ onto X along this path, to obtain a cobordism $X^{\#}$ from $Y \# T^3$ to $Y' \# T^3$; see [Sca15, §7.1]. The choice of path c is suppressed from the notation.

In the case $N = 3$, following the decomposition (5.8) we may write

$$I^{\#,3}(Y, \omega) = \bigoplus_s I^{\#,3}(Y, \omega; s) \tag{8.23}$$

where the direct sum is over homomorphisms $s : H_2(Y; \mathbf{Z}) \to \Gamma \subset \mathbf{Z} \oplus \mathbf{Z}$, with Γ being the sublattice of pairs (a, b) with a and b of the same parity.

In what follows, we write $s = (s_2, s_3)$ for any such homomorphism, where s_2 and s_3 are the projections to the two $\mathbf{Z}$ factors. Further, if $s : H_2(X; \mathbf{Z}) \to \mathbf{Z} \oplus \mathbf{Z}$ is a homomorphism where X is manifold with a submanifold $Y \subset X$, we write $s|_Y$ for the composition of s with the inclusion-induced homomorphism $H_2(Y; \mathbf{Z}) \to H_2(X; \mathbf{Z})$.

The decomposition result for cobordism maps in this setting is as follows.

Theorem 8.24. *Let $(X, w) : (Y, \omega) \to (Y', \omega')$ be a cobordism of pairs with $b_1(X) = 0$ and $b^+(X) > 0$. Then there is a natural decomposition of the cobordism map*

$$I^{\#,3}(X, w) = \sum_s I^{\#,3}(X, w; s),$$

$$I^{\#,3}(X, w; s) : I^{\#,3}(Y, \omega; s|_Y) \to I^{\#,3}(Y', \omega'; s|_{Y'})$$

where the sum is over homomorphisms $s : H_2(X; \mathbf{Z}) \to \mathbf{Z} \oplus \mathbf{Z}$. These maps satisfy:

(i) $I^{\#,3}(X, w; s) = 0$ *for all but finitely many s.*

(ii) *If $I^{\#,3}(X, w; s) \neq 0$, then $s_2(x) + s_3(x) + x \cdot x \equiv 0 \pmod 2$ for all $x \in H_2(X; \mathbf{Z})$, and for any smoothly embedded, connected, orientable surface $\Sigma \subset X$ with non-negative self-intersection and having $[\Sigma]$ non-torsion we have*

$$|s_2([\Sigma]) \pm s_3([\Sigma])| + [\Sigma] \cdot [\Sigma] \leqslant 2g(\Sigma) - 2.$$

(iii) *If* (X,w) *is a composition of two cobordisms* $(X'',w''):(Y,\omega)\to(Y'',\omega'')$ *and* $(X',w'):(Y'',\omega'')\to(Y',\omega')$ *each with* $b_1=0$ *and* $b^+>0$*, then*

$$I^{\#,3}(X',w';s')\circ I^{\#,3}(X'',w'';s'')=\sum I^{\#,3}(X,w;s)$$

where the sum is over $s:H_2(X;\mathbf{Z})\to\mathbf{Z}\oplus\mathbf{Z}$ *such that* $s|_{X'}=s'$ *and* $s|_{X''}=s''$*.*

(iv) *Let* $\widehat{X}=X\#\overline{\mathbb{CP}}^2$ *denote the blowup of* X*, with* e *the exceptional sphere and* E *its Poincaré dual. Let* $\zeta=e^{2\pi i/3}$*. Then for all* $s:H_2(X;\mathbf{Z})\to\mathbf{Z}$ *and* $l,k\in\mathbf{Z}$*:*

$$I^{\#,3}(\widehat{X},w;s+lE_2+kE_3)=\begin{cases}\frac{1}{6}I^{\#,3}(\widehat{X},w;s), & l=\pm1,k=0\\ \frac{1}{3}I^{\#,3}(\widehat{X},w;s), & l=0,k=\pm1\\ 0, & \textit{otherwise}\end{cases}$$

$$I^{\#,3}(\widehat{X},w+e;s+lE_2+kE_3)=\begin{cases}\frac{1}{6}I^{\#,3}(\widehat{X},w;s), & l=\pm1,k=0\\ \frac{1}{3}\zeta^kI^{\#,3}(\widehat{X},w;s), & l=0,k=\pm1\\ 0, & \textit{otherwise}\end{cases}$$

Here E_2 *(resp.* E_3*) is the homomorphism* $H_2(\widehat{X};\mathbf{Z})\to\mathbf{Z}$ *which is pairing with* E *composed with the inclusion of* $\mathbf{Z}$ *into the first factor (resp. second factor) of* $\mathbf{Z}\oplus\mathbf{Z}$*.*

(v) $I^{\#,3}(X,w+a;s)=\zeta^{s_3(a)}I^{\#,3}(X,w;s)$ *for any* $a\in H_2(X;\mathbf{Z})$*.*

This result should be compared to a similar decomposition result in the case of $U(2)$ framed instanton homology, given as Theorem 1.16 in [BS23]. The proof of Theorem 8.24 is largely a consequence of a straightforward adaptation of Theorem 6.26 to the case of cobordisms. This adaptation is carried out in the $U(2)$ case in [BS23]. We only mention some essential points. First, one defines a formal power series in $\Gamma,\Lambda\in H_2(X;\mathbf{R})$ by

$$\mathbb{D}^{\#}_{X,w}(\Gamma_{(2)}+\Lambda_{(3)})=I^{\#,3}(X,w,(1+\frac{1}{3}x_{(2)}+\frac{1}{9}x_{(2)}^2)e^{\Gamma_{(2)}+\Lambda_{(3)}})\tag{8.25}$$

where the notation $I^{\#,3}(X,w,z)$ for $z\in\mathbf{A}^3(X)$ is the cobordism map defined by cutting down via the divisor associated to z. The coefficients of this power series in Γ,Λ are linear maps from $I^{\#,3}(Y,\omega)$ to $I^{\#,3}(Y',\omega')$. The proof of Theorem 6.26 adapts to show that

$$\mathbb{D}^{\#}_{X,w}(\Gamma_{(2)}+\Lambda_{(3)})=e^{\frac{Q(\Gamma)}{2}-Q(\Lambda)}\sum_{i,j}c_{i,j}\zeta^{w\cdot\left(\frac{K_i-K_j}{2}\right)}e^{\frac{\sqrt{3}}{2}(K_i+K_j)\cdot\Gamma+\frac{\sqrt{-3}}{2}(K_i-K_j)\cdot\Lambda}\tag{8.26}$$

The new feature here is that the $c_{i,j}$ are no longer constants, but are instead linear maps $I^{\#,3}(Y,\omega)\to I^{\#,3}(Y',\omega')$. Furthermore, $c_{i,j}$ has coefficients in $\mathbf{Q}[\sqrt{3}]$ with respect to rational bases of the framed instanton groups. Here we view $K_i:H_2(X;\mathbf{Z})\to\mathbf{Z}$; these are characteristic, just as before, and the adjunction inequality is also as stated in (6.27).

The presence of T^3 with its non-trivial bundle in the formation of $X^{\#}$, and the assumption $b_1(X)=0$, guarantee that $X^{\#}$ has the corresponding $U(3)$ simple type condition for all choices of $w \subset X$ (with bundle data over the $I \times T^3$ part in $X^\#$ fixed). By the definition of framed instanton homology, $\beta_2 = \mu_2(x)$ acts as 3, and so the right side of (8.25) is

$$3I^{\#,3}(X, w, e^{\Gamma_{(2)}+\Lambda_{(3)}}).$$

Now for a homomorphism $s = (s_2, s_3) : H_2(X; \mathbf{Z}) \to \mathbf{Z} \oplus \mathbf{Z}$ we define

$$I^{\#,3}(X, w; s) = \begin{cases} \frac{1}{3}\zeta^{w\cdot\left(\frac{K_i - K_j}{2}\right)} c_{i,j} & \text{if } s_2 = \frac{1}{2}(K_i + K_j),\ \ s_3 = \frac{1}{2}(K_i - K_j) \\ 0 & \text{otherwise} \end{cases}$$

The properties listed in Theorem 8.24 are then proved in much the same way as in the $U(2)$ case, using the formula (8.26) and its properties related to the structure theorem; see [BS23] for details. Note that property (iv) follows from a straightforward adaptation of Theorem 6.18, the $N = 3$ blowup formula, to the invariants (8.25).

Remark 8.27. In the $U(2)$ decomposition result of [BS23], the assumption $b^+(X) > 0$ is removed using a trick that involves the trace cobordism of 1-surgery on the $(2, 5)$ torus knot. We expect that the assumption $b^+(X) > 0$ can also be removed from Theorem 8.24.

9 $N = 3$ knot homology and the Alexander polynomial

In this final section, we describe a conjectural relationship between the $U(3)$ instanton knot homology group $KHI^3_*(Y, K)$ introduced in Section 5 and the Alexander polynomial. Throughout this section, K is a knot in an integer homology 3-sphere Y.

There is a $\mathbf{Z}/2$-grading on $KHI^3_*(Y, K)$ defined analogously as in the $U(2)$ case. More generally, there is a relative $\mathbf{Z}/2$-grading on the $U(3)$ sutured instanton homology of any balanced sutured 3-manifold. This is because of the following: $U(3)$ instanton homology of an admissible bundle has a relative $\mathbf{Z}/2$-grading; the operators from which the simultaneous eigenspaces are defined all have even degree; and the excision maps used in the proof of invariance are also homogeneously $\mathbf{Z}/2$-graded. To define an absolute $\mathbf{Z}/2$-grading on $KHI^3_*(Y, K)$, one can use (8.3) for a particular closure of the knot complement. However, the specific choice of convention will not concern us for what follows.

Using the $\mathbf{Z}/2$-grading on $KHI^3_*(Y, K)$ and the decomposition

$$KHI^3_*(Y, K) = \bigoplus_{(a,b)\in\mathcal{C}_{g+1}} KHI^3_*(Y, K; a, b), \tag{9.1}$$

from Section 5 (where g is the Seifert genus of K), which is compatible with the $\mathbf{Z}/2$-grading, we define a Laurent polynomial in two-variables t_2 and t_3 as follows:

$$\Delta^3_{(Y,K)}(t_2, t_3) := \sum_{(a,b)\in\mathcal{C}_{g+1}} \chi(KHI^3_*(Y, K; a, b))t_2^a t_3^b$$

The authors expect that this polynomial is determined by the symmetrized Alexander polynomial $\Delta_{(Y,K)}(t)$ through the following formula:

$$\Delta^3_{(Y,K)}(t_2, t_3) = \pm\Delta_{(Y,K)}(t_2 t_3)\Delta_{(Y,K)}(t_2 t_3^{-1}). \tag{9.2}$$

We present an argument for this relation that is based on some hypotheses which will be made clear momentarily. Let $Z = Z(K)$ be the closed 3-manifold which is the following closure of the sutured manifold associated to the knot:

$$Z = Y \backslash N(K) \cup S^1 \times (F_{1,1} \backslash D^2)$$

Let c, c' be two closed simple curves in the interior of $F_{1,1}$ which generate $H_1(F_{1,1}; \mathbf{Z})$. Then there are tori $\Sigma_1 = S^1 \times c$, $\Sigma_2 = S^1 \times c'$ in Z, and a surface $\Sigma_0 \subset Z$ formed by gluing a Seifert surface for K to $\{pt\} \times \partial F_{1,1}$. (The surface Σ_0 is denoted by $\overline{S}$ at the end of Section 5.) Recall from (5.23) that we have:

$$KHI^3_*(Y, K) = I^3_*(Z, c'|T). \tag{9.3}$$

Consider the 4-manifold $X = S^1 \times Z$. We have tori $\Sigma_3 = S^1 \times c$, $\Sigma_4 = S^1 \times c'$, $\Sigma_5 = S^1 \times \mu$ (in each case the S^1 is external to Z), where μ is a meridian for K. Then

$$H_2(X; \mathbf{Z}) = \bigoplus_{i=0}^{5} \mathbf{Z} \cdot [\Sigma_i]$$

Furthermore, $\Sigma_i \cdot \Sigma_i = 0$ for all i. Note the signature of X is zero. The 4-manifold X is $U(2)$ *strong simple type* in the sense of Muñoz [Muñ00]; this means that

$$D^2_{X,w}(x^2 z) = 4 D^2_{X,w}(z), \qquad D^3_{X,w}(\delta z) = 0 \tag{9.4}$$

for all $z \in \mathbf{A}^2(X) = \mathrm{Sym}^*(H_0(X) \otimes H_2(X)) \otimes \Lambda^* H_1(X)$, $\delta \in H_1(X)$, and $w \in H^2(X; \mathbf{Z})$, where x is a point class. To see that X is $U(2)$ strong simple type, one first shows that (X, w) is $U(2)$ strong simple type for a certain $w \in H^2(X; \mathbf{Z})$. Choose

$$w = \text{P.D.}\left(\Sigma_0 \cup \Sigma_1 \cup \Sigma_2\right).$$

Then w has odd pairing with Σ_i for $3 \leqslant i \leqslant 5$. The strong simple type relations are obtained for $D^2_{X,w}$ through gluing formulas along the Fukaya–Floer homology of $S^1 \times \Sigma_i$ for $3 \leqslant i \leqslant 5$. Here it is key that the surfaces Σ_i for $3 \leqslant i \leqslant 5$ are genus 1, and the Fukaya–Floer homology in the genus 1 case is particularly simple; in particular, the action of a 1-cycle in $S^1 \times \Sigma_i$ on its Fukaya–Floer homology is trivial, and the operator associated to the point class squares to 4 times the identity. As each element of $H_1(X)$ comes from some element of $H_1(\Sigma_i)$ for $3 \leqslant i \leqslant 5$, one obtains the relations

$$D^2_{X,w}(\delta z) = 0$$

for all $\delta \in H_1(X)$ and $z \in \mathbf{A}^2(X) = \mathrm{Sym}^*(H_0(X) \otimes H_2(X)) \otimes \Lambda^* H_1(X)$. Then, strong simple type for one $w \in H^2(X; \mathbf{Z})$ implies the result for all w (see Muñoz [Muñ00]).

Next, as X is $U(2)$ strong simple type, we can apply the structure theorem in this case (again, see Muñoz [Muñ00]). Let $K \in H^2(X; \mathbf{Z})$ be a $U(2)$ basic class for X. Then

$$2g(\Sigma_i) - 2 \geqslant |K \cdot \Sigma_i| + \Sigma_i \cdot \Sigma_i = |K \cdot \Sigma_i|$$

by the adjunction inequality. For $1 \leqslant i \leqslant 5$, we have $g(\Sigma_i) = 1$, and we obtain $K \cdot \Sigma_i = 0$. For $i = 0$, we obtain $|K \cdot \Sigma_0| \leqslant 2g$ where $g = g(\Sigma_0) - 1$ is the Seifert genus of K. Define

$$K_r = 2r\text{P.D.}[\Sigma_5].$$

We conclude that the possible $U(2)$ basic classes of X are the K_r for which $r \in \mathbf{Z}$, $|r| \leqslant g$. Now let $w = \text{P.D.}[\Sigma_4]$. The structure theorem in the $U(2)$ case then reads

$$\widehat{D}^2_{X,w}(e^{\Sigma_0}) = D^2_{X,w}((1+\frac{x}{2})e^{\Sigma_0}) = e^{Q(\Sigma_0)/2}\sum_{r=-g}^{g}(-1)^{(w^2+K_r\cdot w)/2}\beta_r e^{K_r\cdot\Sigma_0}.$$

Witten's conjecture adapted to this case gives $\beta_r = 2^{2+\frac{1}{4}(7\chi(X)+11\sigma(X))}\text{SW}(K_r) = 4\text{SW}_X(K_r)$. Using this, and $Q(\Sigma_0) = 0$, $w \cdot w = 0$, $K_r \cdot w = 0$, we obtain

$$\widehat{D}^2_{X,w}(e^{\Sigma_0}) = 4\sum_{r=-g}^{g}\text{SW}_X(K_r)e^{2r}$$

This invariant can also be expressed as the super trace of a combination of maps induced on the $U(2)$ instanton knot homology $KHI^2_*(Y,K)$ by the cobordism $X' = [0,1] \times Z$ with bundle cyle $w' = [0,1] \times c'$, adorned with the operators $(1+\frac{x}{2})\mu(\Sigma_0)^i$. First, we recall that

$$KHI^2_*(Y,K) = \bigoplus_{j=-g}^{g} KHI^2_*(Y,K;j)$$

where $KHI_*(Y,K)$ is the generalized eigenspace of $\mu(pt)$ acting on $I^2_*(Z,c')$ with eigenvalue 2, and $KHI^2_*(Y,K;j)$ is the simultaneous generalized eigenspace of $(\mu(pt),\mu(\Sigma_0))$ acting on $I^2_*(Z,c')$ with eigenvalues $(2,2j)$. We compute:

$$\begin{aligned}\widehat{D}^2_{X,w}(e^{\Sigma_0}) &= 2\sum_{i=0}^{\infty}\frac{1}{i!}\text{tr}_s\left(I^2(X',w',(1+\frac{x}{2})\mu(\Sigma_0)^i)\right)\\ &= 2\sum_{j=-g}^{g}\sum_{i=0}^{\infty}\frac{1}{i!}\chi(KHI_*(Y,K;j))2(2j)^i = 4\sum_{j=-g}^{g}A_j e^{2j}\end{aligned}$$

where A_j is the coefficient of t^j in $\Delta_{(Y,K)}(t)$. Here, in the last equality, we have used that the graded Euler characteristic of $KHI_*(Y,K)$ is the Alexander polynomial [KM10a, Lim10]. The "2" appearing outside the summands in the middle expressions comes from a gluing factor (see the discussion [KM11a, §5.2]). In particular, we obtain

$$\text{SW}_X(K_r) = A_r. \tag{9.5}$$

We then repeat this analysis in the $U(3)$ setting. Assume X is $U(3)$ simple type (we predict this is true, but we cannot adapt the above argument in the $U(2)$ case; see Remark 6.28). Mariño and Moore [MM98] conjecture that the basic classes in the $U(3)$ structure theorem are the same as the $U(2)$ basic classes (see also [DX20, Conjecture 7.2]). The $U(3)$ structure theorem then gives the following:

$$\widehat{D}^3_{X,w}(e^{s_2(\Sigma_0)_{(2)}+s_3(\Sigma_0)_{(3)}}) = \sum_{i,j} c_{i,j} e^{s_2\sqrt{3}(i+j)+s_3\sqrt{-3}(i-j)}. \tag{9.6}$$

The conjecture also states that the constants $c_{i,j}$ are given by

$$\begin{aligned} c_{i,j} &= 2^{\chi(X)+\frac{3}{2}\sigma(X)+\frac{1}{2}K_i\cdot K_j} 3^{2+\frac{7}{4}\chi(X)+\frac{11}{4}\sigma(X)} \mathrm{SW}_X(K_i)\mathrm{SW}_X(K_j) \\ &= 9\mathrm{SW}_X(K_i)\mathrm{SW}_X(K_j). \end{aligned} \tag{9.7}$$

We compute that the left side of (9.6) is also equal to the following (where the "3" on the outside of the first sum comes from a gluing factor analogous to the $N = 2$ case):

$$\begin{aligned} &3\sum_{k,l=0}^{\infty} \frac{1}{k!l!}\mathrm{tr}_s\left(I^3(X', w', (1+\frac{x_{(2)}}{3}+\frac{x_{(2)}^2}{9})(s_2\mu_2(\Sigma_0))^k(s_3\mu_3(\Sigma_0))^l)\right) \\ &\qquad = 3\sum_{(a,b)\in\mathcal{C}_{g+1}} \sum_{k,l=0}^{\infty} \frac{1}{k!l!}\chi(KHI^3_*(Y,K;a,b))3(\sqrt{3}a)^k(\sqrt{-3}b)^l \\ &\qquad = 9\sum_{(a,b)\in\mathcal{C}_{g+1}}^{g} A_{a,b}e^{\sqrt{3}as_2+\sqrt{-3}bs_3} \end{aligned}$$

where $A_{a,b}$ is the coefficient of $t_2^a t_3^b$ in $\Delta^3_{(Y,K)}(t_3, t_3)$. By (9.5), (9.6) and (9.7), we get:

$$A_{a,b} = c_{(a+b)/2,(a-b)/2} = A_{(a+b)/2}A_{(a-b)/2}.$$

This establishes (9.2) under the assumptions stated above, apart from an overall sign $\pm$, which is determined by conventions that are not discussed here.

A question arises regarding the extent to which the relationship (9.2), which is at the level of graded Euler characteristics, might hold at the level of Floer homologies. For example, in the special case that the $U(2)$ and $U(3)$ instanton knot homology groups of (Y, K) are supported in even gradings, (9.2) implies a vector space isomorphism

$$KHI^3_*(Y,K) \cong KHI^2_*(Y,K) \otimes KHI^2_*(Y,K). \tag{9.8}$$

Thus, along the same lines following (8.19), we are led to ask: is there a knot for which the isomorphism (9.8) does not hold?

Remark 9.9. The 4-manifold $X = S^1 \times Z(K)$ is an instance of Fintushel–Stern's knot surgery on a standard T^2 inside the 4-torus T^4, and the relationship (9.5) is the same kind that is established in [FS98] between Seiberg–Witten invariants and the Alexander polynomial (see [Ni17] for a more general statement, relevant to our case). One approach to proving (9.2) is to establish Fintushel–Stern knot surgery formulas in the setting of Donaldson invariants, of type $U(2)$ and $U(3)$, removing the dependency of the above argument on conjectural relationships to Seiberg–Witten theory.

Remark 9.10. The $U(2)$ instanton knot homology group $KHI^2_*(Y, K)$ is isomorphic to a version of singular instanton homology where one takes the connected sum of (Y, K) with the Hopf link in S^3, uses a bundle associated to an arc connecting the two resulting link components, and the singular condition is that the holonomy of a connection along shrinking meridians limits to an element in $U(2)$ conjugate to $\text{diag}(i, -i)$. See [KM11a, §5.4].

The $U(3)$ instanton knot homology $KHI^3_*(Y, K)$ is isomorphic to a version of $U(3)$ singular instanton homology as developed in [KM11b], using a similar description as above, but with the singular condition that the holonomy of a connection along shrinking meridians limits to an element in $U(3)$ conjugate to $\text{diag}(1, \zeta, \zeta^2)$. A similar application of excision as in [KM11a] can be used to give this isomorphism.

A natural question is whether the relationship between Khovanov homology and the instanton group $KHI^2_*(S^3, K)$ established in [KM11a] has a counterpart in the setting of $U(3)$ (or more generally, $U(N)$), and if so, what quantum knot homology theory plays the role of Khovanov homology in this setting.

References

[AB83] M. F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. MR702806 ↑14, 15

[APS75a] M. F. Atiyah, V. K. Patodi, and I. M. Singer, *Spectral asymmetry and Riemannian geometry. I*, Math. Proc. Cambridge Philos. Soc. **77** (1975), 43–69. MR397797 ↑58

[APS75b] ______, *Spectral asymmetry and Riemannian geometry. II*, Math. Proc. Cambridge Philos. Soc. **78** (1975), no. 3, 405–432. MR397798 ↑58

[Bar94] V. Yu. Baranovskiĭ, *The cohomology ring of the moduli space of stable bundles with odd determinant*, Izv. Ross. Akad. Nauk Ser. Mat. **58** (1994), no. 4, 204–210. MR1307064 ↑6

[BD95] Peter Braam and Simon Donaldson, *Fukaya-Floer homology and gluing formulae for polynomial invariants*, The Floer memorial volume, 1995, pp. 257–281. MR1362830 ↑36

[Bor53] Armand Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207. MR51508 ↑60

[BS23] John A. Baldwin and Steven Sivek, *Instantons and L-space surgeries*, J. Eur. Math. Soc. (JEMS) **25** (2023), no. 10, 4033–4122. MR4634689 ↑4, 54, 60, 61, 62, 63

[CDX17] Lucas Culler, Aliakbar Daemi, and Yi Xie, *Surgery, Polygons and $SU(N)$-Floer Homology*, arXiv e-prints (2017Dec), arXiv:1712.09910, available at `1712.09910`. ↑59, 61

[Cul14] Lucas Howard Culler, *The Blowup Formula for Higher Rank Donaldson Invariants*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–Massachusetts Institute of Technology. MR3279020 ↑3, 42

[Dae] Aliakbar Daemi, *Abelian Gauge Theory, Knots and Odd Khovanov Homology*, available at `1508.07650`. ↑60

[Don02] S. K. Donaldson, *Floer homology groups in Yang-Mills theory*, Cambridge Tracts in Mathematics, vol. 147, Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick. MR1883043 ↑55, 57

[Don99] ______, *Lefschetz pencils on symplectic manifolds*, J. Differential Geom. **53** (1999), no. 2, 205–236. MR1802722 ↑51

[DX20] Aliakbar Daemi and Yi Xie, *Sutured manifolds and polynomial invariants from higher rank bundles*, Geom. Topol. **24** (2020), no. 1, 49–178. MR4080482 ↑2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 26, 27, 30, 31, 36, 39, 40, 41, 42, 44, 50, 52, 53, 58, 66

[Ear97] Richard Earl, *The Mumford relations and the moduli of rank three stable bundles*, Compositio Math. **109** (1997), no. 1, 13–48. MR1473604 ↑6, 15, 18

[Eis23] Mike Miller Eismeier, *Equivariant instanton homology* (2023), available at `arXiv:1907.01091`. ↑59

[EK04] Richard Earl and Frances Kirwan, *Complete sets of relations in the cohomology rings of moduli spaces of holomorphic bundles and parabolic bundles over a Riemann surface*, Proc. London Math. Soc. (3) **89** (2004), no. 3, 570–622. MR2107008 ↑6, 15

[Eli04] Yakov Eliashberg, *A few remarks about symplectic filling*, Geom. Topol. **8** (2004), 277–293. MR2023279 ↑51

[ET98] Yakov M. Eliashberg and William P. Thurston, *Confoliations*, University Lecture Series, vol. 13, American Mathematical Society, Providence, RI, 1998. MR1483314 ↑51

[Etn04] John B. Etnyre, *On symplectic fillings*, Algebr. Geom. Topol. **4** (2004), 73–80. MR2023278 ↑51

[Flo95] A. Floer, *Instanton homology and Dehn surgery*, The Floer memorial volume, 1995, pp. 77–97. MR1362823 ↑7, 61

[FS98] Ronald Fintushel and Ronald J. Stern, *Knots, links, and* 4*-manifolds*, Invent. Math. **134** (1998), no. 2, 363–400. MR1650308 ↑66

[Fuk92] Kenji Fukaya, *Floer homology for oriented* 3*-manifolds*, Aspects of low-dimensional manifolds, 1992, pp. 1–92. MR1208307 ↑36

[Gab83] David Gabai, *Foliations and the topology of* 3*-manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503. MR723813 ↑29, 32, 34, 51

[GS99] Robert E. Gompf and András I. Stipsicz, 4*-manifolds and Kirby calculus*, Graduate Studies in Mathematics, vol. 20, American Mathematical Society, Providence, RI, 1999. MR1707327 ↑53

[HS41] H. Hopf and H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment. Math. Helv. **13** (1941), 240–251. MR6546 ↑57

[Juh06] András Juhász, *Holomorphic discs and sutured manifolds*, Algebr. Geom. Topol. **6** (2006), 1429–1457. MR2253454 ↑3, 29

[Juh08] András Juhász, *Floer homology and surface decompositions*, Geom. Topol. **12** (2008), no. 1, 299–350. MR2390347 ↑32, 33, 34

[Kir78] Rob Kirby, *Problems in low dimensional manifold theory*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, 1978, pp. 273–312. MR520548 ↑2

[Kir92] Frances Kirwan, *The cohomology rings of moduli spaces of bundles over Riemann surfaces*, J. Amer. Math. Soc. **5** (1992), no. 4, 853–906. MR1145826 ↑15, 18

[KM04] P. B. Kronheimer and T. S. Mrowka, *Witten's conjecture and property P*, Geom. Topol. **8** (2004), 295–310. MR2023280 ↑51

[KM07] Peter Kronheimer and Tomasz Mrowka, *Monopoles and three-manifolds*, New Mathematical Monographs, vol. 10, Cambridge University Press, Cambridge, 2007. MR2388043 ↑51

[KM10a] Peter Kronheimer and Tom Mrowka, *Instanton Floer homology and the Alexander polynomial*, Algebr. Geom. Topol. **10** (2010), no. 3, 1715–1738. MR2683750 ↑4, 65

[KM10b] Peter Kronheimer and Tomasz Mrowka, *Knots, sutures, and excision*, J. Differential Geom. **84** (2010), no. 2, 301–364. MR2652464 ↑2, 3, 7, 9, 10, 26, 27, 28, 30, 32, 33, 35, 55

[KM11a] P. B. Kronheimer and T. S. Mrowka, *Khovanov homology is an unknot-detector*, Publ. Math. Inst. Hautes Études Sci. **113** (2011), 97–208. MR2805599 ↑54, 60, 65, 67

[KM11b] Peter Kronheimer and Tomasz Mrowka, *Knot homology groups from instantons*, J. Topol. **4** (2011), no. 4, 835–918. MR2860345 (2012m:57060) ↑4, 7, 45, 54, 55, 56, 58, 60, 67

[KM13] P. B. Kronheimer and T. S. Mrowka, *Gauge theory and Rasmussen's invariant*, J. Topol. **6** (2013), no. 3, 659–674. MR3100886 ↑45

[KM18] Peter B. Kronheimer and Tomasz S. Mrowka, *Knots, three-manifolds and instantons*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, 2018, pp. 607–634. MR3966740 ↑2

[KM93] P. B. Kronheimer and T. S. Mrowka, *Gauge theory for embedded surfaces. I*, Topology **32** (1993), no. 4, 773–826. MR1241873 ↑45

[KM95] ______, *Embedded surfaces and the structure of Donaldson's polynomial invariants*, J. Differential Geom. **41** (1995), no. 3, 573–734. MR1338483 ↑5, 36, 50

[KN98] A. D. King and P. E. Newstead, *On the cohomology ring of the moduli space of rank* 2 *vector bundles on a curve*, Topology **37** (1998), no. 2, 407–418. MR1489212 ↑6

[Kro05] P. B. Kronheimer, *Four-manifold invariants from higher-rank bundles*, J. Differential Geom. **70** (2005), no. 1, 59–112. MR2192061 ↑3, 55, 56, 60

[Lim10] Yuhan Lim, *Instanton homology and the Alexander polynomial*, Proc. Amer. Math. Soc. **138** (2010), no. 10, 3759–3768. MR2661575 ↑4, 65

[Muñ00] Vicente Muñoz, *Basic classes for four-manifolds not of simple type*, Comm. Anal. Geom. **8** (2000), no. 3, 653–670. MR1775702 ↑5, 45, 47, 49, 50, 64

[Muñ99] ______, *Ring structure of the Floer cohomology of* $\Sigma \times \mathbf{S}^1$, Topology **38** (1999), no. 3, 517–528. MR1670396 ↑6, 9, 11

[MM98] Marcos Mariño and Gregory Moore, *The Donaldson-Witten function for gauge groups of rank larger than one*, Comm. Math. Phys. **199** (1998), no. 1, 25–69. MR1660219 ↑3, 4, 5, 65

[Mor84] John W. Morgan, *The Smith conjecture*, The Smith conjecture (New York, 1979), 1984, pp. 3–6. MR758460 ↑3

[Ni17] Yi Ni, *Fintushel-Stern knot surgery in torus bundles*, J. Topol. **10** (2017), no. 1, 164–177. MR3653065 ↑66

[OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic triangle invariants and the topology of symplectic four-manifolds*, Duke Math. J. **121** (2004), no. 1, 1–34. MR2031164 ↑5

[Sca15] Christopher W. Scaduto, *Instantons and odd Khovanov homology*, J. Topol. **8** (2015), no. 3, 744–810. MR3394316 ↑4, 35, 54, 55, 56, 59, 61

[Siv15] Steven Sivek, *Donaldson invariants of symplectic manifolds*, Int. Math. Res. Not. IMRN **6** (2015), 1688–1716. MR3340371 ↑5

[Smi01] Ivan Smith, *Lefschetz pencils and divisors in moduli space*, Geom. Topol. **5** (2001), 579–608. MR1833754 ↑51

[ST95] Bernd Siebert and Gang Tian, *Recursive relations for the cohomology ring of moduli spaces of stable bundles*, Turkish J. Math. **19** (1995), no. 2, 131–144. MR1349566 ↑6

[Zag95] Don Zagier, *On the cohomology of moduli spaces of rank two vector bundles over curves*, The moduli space of curves (Texel Island, 1994), 1995, pp. 533–563. MR1363070 ↑6, 15, 16, 17