Math 461: Algebra Midterm - preparation

Justify all steps, or else I'll call a United flight attendant.

Name:

Problem 1. Let A be a domain. Show that A is a field if and only if A has finitely many ideals.

Problem 2. Give an example of a polynomial in $\mathbb{R}[x]$ of positive degree that has no roots, but is not irreducible. *Bonus points*: What is the smallest degree that one such polynomial can have?

Problem 3. Prove that the quotient ${\mathbb{Z}_4[x]}/{(x,\overline{2})}$ is isomorphic to $\mathbb{Z}_2$.

Problem 4. Show that in $Z_5[x]$, the polynomial $x^3 + 2x^2 + 3$ is irreducible. Is the quotient

$$Z_5[x]/(x^3 + 2x^2 + 3)$$

a field?

Problem 5. (Ex. 20 p. 248) Let $a, b \in \mathbb{Z}$. Show that $(a) + (b) = \mathbb{Z}$ if and only if a and b are coprime.

Problem 6. (Ex. 15 p. 262) Let F be a commutative ring with unity. Suppose that in F every proper ideal is prime. Prove that F is a field.

Problem 7. Find the GCD of $x^6 + x^5 + x^4 - x^2 - x - 1$ and $x^4 + x$.

Problem 8. Let A be a commutative ring. Show that if $A[x]$ is a PID, then A is a field.

Problem 9. How many ideals with less than 10 elements does $\mathbb{Z}$ have ?

Problem 10. Consider the ideal $J = (x+1, x-1)$ in $\mathbb{Z}[x]$. Is it proper? Is it prime? Is it maximal?

Cheat sheet allowed