Math 461 - Finale

Upon request i am circulating a preparation for the final. I will *not* have time to post solutions to these exercises, but you are strong enough to figure them out.

Name:

Problem 1. Let G be a group in which all elements have order ≤ 2. Prove that G is Abelian.

Problem 2. Prove any way you want that every group with 4 elements is Abelian. Show that the smallest cardinality of a non-Abelian group is six.

Problem 3. Prove that in ${}^{\mathbb{Q}}\!/\!_{\mathbb{Z}}$ every element has finite period.

Problem 4. Let G and H be commutative rings. Consider the subring $\{0\}$ of G. Show that $\frac{G \times H}{\{0\} \times H}$ is isomorphic to G.

Problem 5. How many roots does $x^4 + 1$ have in $Z_5[x]$?
Bonus points: Is $x^4 + 1$ irreducible?

Problem 6. Let $A = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous}\}$. Show that A is a ring, with the operation defined pointwise. (With this I mean that "$f + g$" is the map that sends any $x \in \mathbb{R}$ to $f(x) + g(x)$; similarly, "$f \cdot g$" is the map that sends x to $f(x)g(x)$.)

Bonus point: Show that $B = \{f : \mathbb{R} \to \mathbb{R} \text{ continuous, such that } f(3) = 0\}$ is an ideal of A.

Problem 7. Does it make sense to speak of a "GCD" of two polynomials in $\mathbb{Z}[x, y]$? How would you define it?

Problem 8. Find a GCD of $x^3 - 1$ and $x^5 - 1$ in $\mathbb{R}[x]$. Is the GCD unique, or can you find another one?

Problem 9. Prove that $\sqrt{\pi} + 1$ is not algebraic over Q.

Problem 10. Find $[Q(\sqrt[5]{7} - 1) : Q]$.