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MAT 461D

An example of non-UFD ring, and

∂ IRREDUCIBLE $\not\Rightarrow$ (∂) PRIME.

Let $A = \mathbb{Z}[\sqrt{-5}] = \{ z \in \mathbb{C} \text{ s.t. } z = n + m \cdot i \cdot \sqrt{5},\ n, m \in \mathbb{Z} \}$

- A is a DOMAIN.
  ∴ It is, in fact, a subring of $\mathbb{Z}[i]$, which is a domain.
- Inside A, we have

$$2 \cdot 3 = 6 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$$

- (2) is NOT prime.
  ∴ In fact, $(1+\sqrt{5}\,i) \notin (2)$, for otherwise, we would have
  $1+\sqrt{5}\,i = 2(n+mi\sqrt{5})$, with $m \in \mathbb{Z}$ $\Rightarrow \begin{cases} 2n = 1 \Rightarrow \text{absurd, } 1 \text{ not even,} \\ 2m = 1 \end{cases}$
  Similarly, $(1-\sqrt{5}\,i)$ is not a multiple of 2 either.
  So $(1+\sqrt{5}\,i)(1-\sqrt{5}\,i) \in (2)$, but each factor doesn't.
- 2 is IRREDUCIBLE.
  ∴ The reason is that a product $(a+bi\sqrt{5})(c+di\sqrt{5})$ yields an integer only if the "$i\sqrt{5}$" part cancels out. Formally the product is $(ac-5bd) + (ad+bc) \cdot i\sqrt{5}$, which belongs to $\mathbb{Z}$ iff $ad = -bc$; which means that up to a scalar, $c = a$ and $d = -b$.
  But $(a+bi\sqrt{5})(a-bi\sqrt{5}) = a^2 + 5b^2$ is STRICTLY LARGER THAN 2 unless $b = 0$! So there's no way to write 2 as product of two elements in A, unless they're in $\mathbb{Z}$. $\square$

NOTE. Similarly, 3 is irreducible, but not prime. Also $(1 \pm \sqrt{5}\,i)$ are irreducible. So
$\Rightarrow$ A is NOT a UFD.