SECOND EDITION

Elementary Linear Programming with Applications

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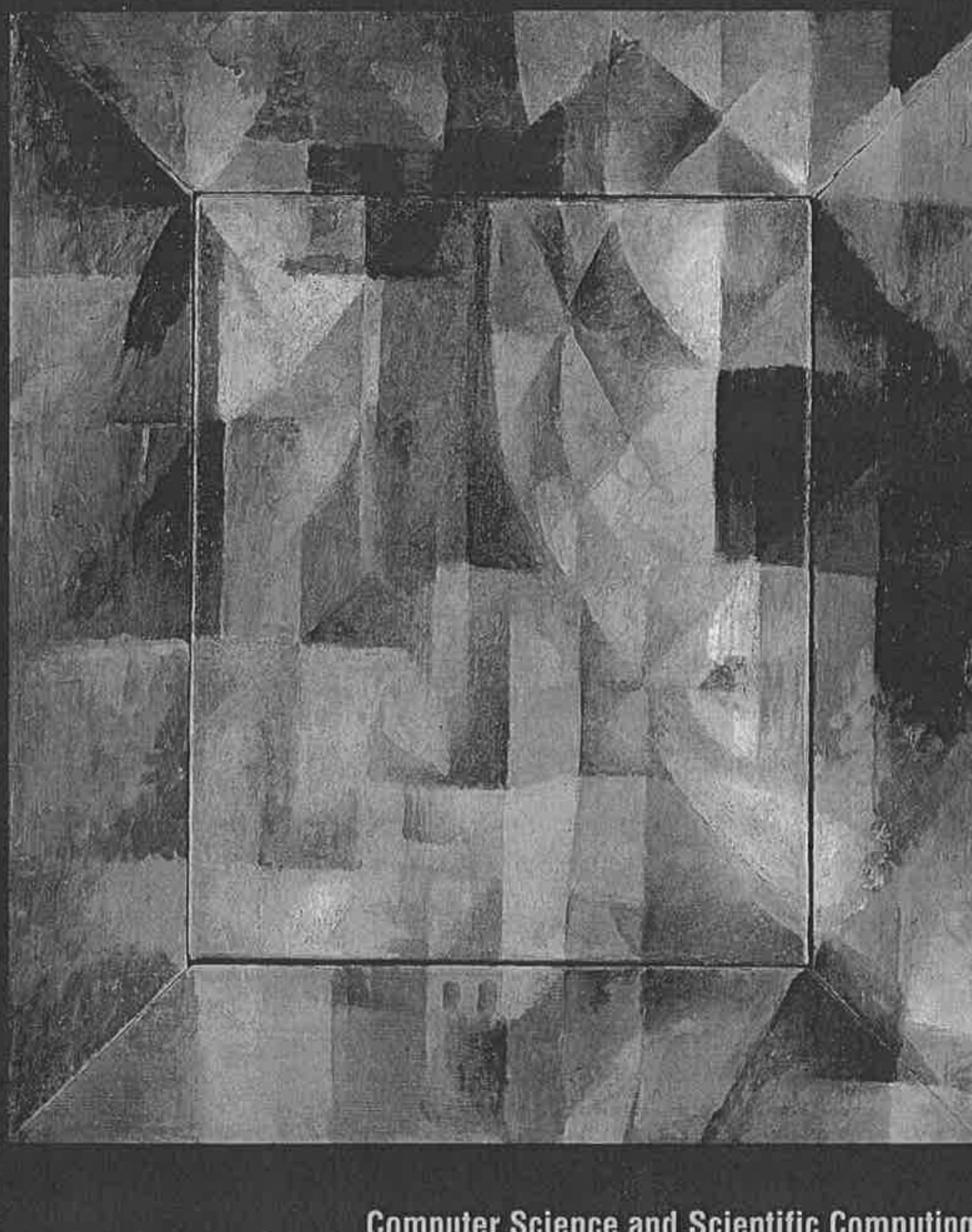

Computer Science and Scientific Computing

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The Simplex Method

IN THIS CHAPTER we describe an elementary version of the method that can be used to solve a linear programming problem systematically. In Chapter 1 we developed the algebraic and geometric notions that allowed us to characterize the solutions to a linear programming problem. However, for problems of more than three variables, the characterization did not lead to a practical method for actually finding the solutions. We know that the solutions are extreme points of the set of feasible solutions. The method that we present determines the extreme points in the set of feasible solutions in a particular order that allows us to find an optimal solution in a small number of trials. We first consider problems in standard form because when applying the method to these problems it is easy to find a starting point. The second section discusses a potential pitfall with the method. However, the difficulty rarely arises and has almost never been found when solving practical problems. In the third section, we extend the method to arbitrary linear programming problems by developing a way of constructing a starting point.

2.1 THE SIMPLEX METHOD FOR PROBLEMS IN STANDARD FORM

We already know from Section 1.5 that a linear programming problem in canonical form can be solved by finding all the basic solutions, discarding those that are not feasible, and finding an optimal solution among the remaining. Since this procedure can still be a lengthy one, we seek a more efficient method for solving linear programming problems. The simplex algorithm is such a method; in this section we shall describe and carefully illustrate it. Even though the method is an algebraic one, it is helpful to examine it geometrically.

Consider a linear programming problem in standard form

$$\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{1}$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{2}$$

$$\mathbf{x} \geq \mathbf{0}, \tag{3}$$

where $\mathbf{A} = [a_{ij}]$ is an $m \times n$ matrix and

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

In this section we shall make the additional assumption that $\mathbf{b} \geq \mathbf{0}$. In Section 2.3 we will describe a procedure for handling problems in which $\mathbf{b}$ is not nonnegative.

We now transform each of the constraints in (2) into an equation by introducing a slack variable. We obtain the canonical form of the problem, namely

$$\text{Maximize} \quad z = \mathbf{c}^{\mathrm{T}}\mathbf{x} \tag{4}$$

subject to

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{5}$$

$$\mathbf{x} \geq \mathbf{0}, \tag{6}$$

where in this case $\mathbf{A}$ is the $m \times (n + m)$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix},$$

and $\mathbf{b}$ is as before.

Recall from Section 1.5 that a basic feasible solution to the canonical form of the problem (4), (5), (6) is an extreme point of the convex set S' of all feasible solutions to the problem.

DEFINITION. Two distinct extreme points in S' are said to be **adjacent** if as basic feasible solutions they have all but one basic variable in common. △

EXAMPLE 1. Consider Example 2 of Section 1.5 and especially Table 1.5 in that example. The extreme points $(0, 0, 8, 15)$ and $(0, 4, 0, 3)$ are adjacent, since the basic variables in the first extreme point are u and v and the basic variables in the second extreme point are y and v. In fact, the only extreme point that is not adjacent to $(0, 0, 8, 15)$ is $(\frac{3}{2}, \frac{5}{2}, 0, 0)$. △

The simplex method developed by George B. Dantzig in 1947 is a method that proceeds from a given extreme point (basic feasible solution) to an adjacent extreme point in such a way that the value of the objective function increases or, at worst, remains the same. The method proceeds until we either obtain an optimal solution or find that the given problem has no finite optimal solution. The simplex algorithm consists of two steps: (1) a way of finding out whether a given basic feasible solution is an optimal solution and (2) a way of obtaining an adjacent basic feasible solution with the same or larger value for the objective function. In actual use, the simplex method does not examine every basic feasible solution; it checks only a relatively small number of them. However, examples have been given in which a large number of basic feasible solutions have been examined by the simplex method.

We shall demonstrate parts of our description of the simplex method on the linear programming problem in Example 1 of Section 1.1. The associated canonical form of the problem was described in Example 4 of Section

1.2. In this form it is:

$$\text{Maximize} \quad z = 120x + 100y \tag{7}$$

subject to

$$\left.\begin{aligned} 2x + 2y + u \phantom{{}+v} &= 8 \\ 5x + 3y \phantom{{}+u} + v &= 15 \end{aligned}\right\} \tag{8}$$

$$x \geq 0, \quad y \geq 0, \quad u \geq 0, \quad v \geq 0. \tag{9}$$

The Initial Basic Feasible Solution

To start the simplex method, we must find a basic feasible solution. The assumption that $\mathbf{b} \geq \mathbf{0}$ allows the following procedure to work. If it is not true that $\mathbf{b} \geq \mathbf{0}$, another procedure (discussed in Section 2.3) must be used. We take all the nonslack variables as nonbasic variables; that is, we set all the nonslack variables in the system $\mathbf{Ax} = \mathbf{b}$ equal to zero. The basic variables are then just the slack variables. We have

$$x_1 = x_2 = \cdots = x_n = 0 \quad \text{and} \quad x_{n+1} = b_1, \quad x_{n+2} = b_2, \ldots, x_{n+m} = b_m.$$

This is a feasible solution, since $\mathbf{b} \geq \mathbf{0}$; and it is a basic solution, since $(n + m) - m = n$ of the variables are zero.

In our example, we let

$$x = y = 0.$$

Solving for u and v, we obtain

$$u = 8, \quad v = 15.$$

The initial basic feasible solution constructed by this method is $(0, 0, 8, 15)$. The basic feasible solution yields the extreme point $(0, 0)$ in Figure 1.14 (Section 1.4).

It is useful to set up our example and its initial basic feasible solution in tabular form. To do this, we write (7) as

$$-120x - 100y + z = 0, \tag{10}$$

where z is now viewed as another variable. The **initial tableau** is now formed (Tableau 2.1). At the top we list the variables x, y, u, v, and z as labels on the corresponding columns. The last row, called the **objective row**, is Equation (10). The constraints (8) are on the first two rows. Along the left side of each row we indicate which variable is basic in the corresponding equation. Thus, in the first equation u is the basic variable, and v is the basic variable in the second equation.

Tableau 2.1

	x	y	u	v	z	
u	2	2	1	0	0	8
v	5	3	0	1	0	15
	-120	-100	0	0	1	0

In the tableau, a basic variable has the following properties:

1. It appears in exactly one equation and in that equation it has a coefficient of $+1$.
2. The column that it labels has all zeros (including the objective row entry) except for the $+1$ in the row that is labeled by the basic variable.
3. The value of a basic variable is the entry in the same row in the rightmost column.

The initial tableau for the general problem (4), (5), (6) is shown in Tableau 2.2. The value of the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n + 0 \cdot x_{n+1} + \cdots + 0 \cdot x_{n+m}$$

for the initial basic feasible solution is

$$z = c_1 \cdot 0 + c_2 \cdot 0 + \cdots + c_n \cdot 0 + 0 \cdot b_1 + 0 \cdot b_2 + \cdots + 0 \cdot b_m = 0.$$

Notice that the entry in the last row and rightmost column is the value of the objective function for the initial basic feasible solution.

Tableau 2.2

	x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	z	
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	0	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2n}	0	1	$\cdots$	0	0	b_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	0	b_m
	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0	0	$\cdots$	0	1	0

In our example we have

$$z = 120 \cdot 0 + 100 \cdot 0 + 0 \cdot 8 + 0 \cdot 15 = 0.$$

At this point the given linear programming problem has been transformed to the initial tableau. This tableau displays the constraints and objective function along with an initial basic feasible solution and the corresponding value of the objective function for this basic feasible solution. We are now ready to describe the steps in the simplex method that are used repeatedly to create a sequence of tableaux, terminating in a tableau that yields an optimal solution to the problem.

Checking an Optimality Criterion

We shall now turn to the development of a criterion that will determine whether the basic feasible solution represented by a tableau is, in fact, optimal. For our example we can increase the value of z from its value of 0 by increasing any one of the nonbasic variables having a positive coefficient from its current value of 0 to some positive value. For our example,

$$z = 120x + 100y + 0 \cdot u + 0 \cdot v,$$

so that z can be increased by increasing either x or y.

For an arbitrary tableau, if we write the objective function so that the coefficients of the basic variables are zero, we then have

$$z = \sum_{\text{nonbasic}} d_j x_j + \sum_{\text{basic}} 0 \cdot x_i, \tag{11}$$

where the d_j's are the negatives of the entries in the objective row of the tableau. We see that (11) has some terms with positive coefficients if and only if the objective row has negative entries under some of the columns labeled by nonbasic variables. Now the value of z can be increased by increasing the value of any nonbasic variable with a negative entry in the objective row from its current value of 0. If this is done, then some basic variable must be set to zero since the number of basic variables is to remain unchanged. Setting this basic variable to zero will not change the value of the objective function since the coefficient of the basic variable was zero. We summarize this discussion by stating the following optimality criterion for testing whether a feasible solution shown in a tableau is an optimal solution.

Optimality Criterion. If the objective row of a tableau has zero entries in the columns labeled by basic variables and no negative entries in the columns labeled by nonbasic variables, then the solution represented by the tableau is optimal.

As soon as the optimality criterion has been met, we can stop our computations, for we have found an optimal solution.

Selecting the Entering Variable

Suppose now that the objective row of a tableau has negative entries in the labeled columns. Then the solution shown in the tableau is not optimal, and some adjustment of the values of the variables must be made.

The simplex method proceeds from a given extreme point (basic feasible solution) to an *adjacent* extreme point in such a way that the objective function increases in value. From the definition of adjacent extreme point, it is clear that we reach such a point by increasing a single variable from zero to a positive value and decreasing a variable with a positive value to zero. The largest increase in z per unit increase in a variable occurs for the most negative entry in the objective row. We shall see below that, if the feasible set is bounded, there is a limit on the amount by which we can increase a variable. Because of this limit, it may turn out that a larger increase in z may be achieved by *not* increasing the variable with the most negative entry in the objective row. However, this rule is most commonly followed because of its computational simplicity. Some computer implementations of the simplex algorithm provide other strategies for choosing the variable to be increased, including one as simple as choosing the first negative entry. Another compares increases in the objective function for several likely candidates for the entering variable. In Tableau 2.1, the most negative entry, -120, in the objective row occurs under the x column, so that x is chosen to be the variable to be increased from zero to a positive value. The variable to be increased is called the **entering variable**, since in the next iteration it will become a basic variable; that is, it will *enter* the set of basic variables. If there are several possible entering variables, choose one. (This situation will occur when the most negative entry in the objective row occurs in more than one column.) Now an increase in one variable must be accompanied by a decrease in some of the other variables to maintain a solution to $\mathbf{Ax} = \mathbf{b}$.

Choosing the Departing Variable

Solving (8) for the basic variables u and v, we have

$$\begin{aligned} u &= \ \ 8 - 2x - 2y \\ v &= 15 - 5x - 3y. \end{aligned}$$

We increase only x and keep y at zero. We have

$$\left.\begin{aligned} u &= \ \ 8 - 2x \\ v &= 15 - 5x \end{aligned}\right\}, \tag{12}$$

which shows that as x increases both u and v decrease. By how much can we increase x? It can be increased until either u or v becomes negative.

That is, from (9) and (12) we have

$$0 \le u = 8 - 2x$$
$$0 \le v = 15 - 5x.$$

Solving these inequalities for x, we find

$$2x \le 8 \quad \text{or} \quad x \le 8/2 = 4$$

and

$$5x \le 15 \quad \text{or} \quad x \le 15/5 = 3.$$

We see that we cannot increase x by more than the smaller of the two ratios $8/2$ and $15/5$. Letting $x = 3$, we obtain a new feasible solution,

$$x = 3, \quad y = 0, \quad u = 2, \quad v = 0.$$

In fact, this is a basic feasible solution, and it was constructed to be adjacent to the previous basic feasible solution, since only one variable changed from basic to nonbasic. The new basic variables are x and u; the nonbasic variables are y and v. The objective function now has the value

$$z = 120 \cdot 3 + 100 \cdot 0 + 0 \cdot 2 + 0 \cdot 0 = 360,$$

which is a considerable improvement over the previous value of zero.

The new basic feasible solution yields the extreme point $(3, 0)$ in Figure 1.14, and it is adjacent to $(0, 0)$. In the new basic feasible solution to our example, we have the variable $v = 0$. It is no longer a basic variable because it is zero, and it is called a **departing variable** since it has *departed* from the set of basic variables. The column of the entering variable is called the **pivotal column**; the row that is labeled with the departing variable is called the **pivotal row**.

We now examine more carefully the selection of the departing variable. Recall that the ratios of the rightmost column entries to the corresponding entries in the pivotal column were determined by how much we could increase the entering variable (x in our example). These ratios are called **θ-ratios**. The smallest nonnegative θ-ratio is the largest possible value for the entering variable. The basic variable labeling the row where the smallest nonnegative θ-ratio occurs is the departing variable, and the row is the pivotal row. In our example,

$$\min\{8/2, 15/5\} = 3,$$

and the second row in Tableau 2.1 is the pivotal row.

If the smallest nonnegative θ-ratio is not chosen, then the next basic solution is not feasible. Suppose we had chosen u as the departing variable by choosing the θ-ratio as 4. Then $x = 4$, and from (12) we have

$$u = 8 - 2 \cdot 4 = 0$$
$$v = 15 - 5 \cdot 4 = -5,$$

and the next basic solution is

$$x = 4, \quad y = 0, \quad u = 0, \quad v = -5,$$

which is not feasible.

In the general case, we have assumed that the rightmost column will contain only nonnegative entries. However, the entries in the pivotal column may be positive, negative, or zero. Positive entries lead to nonnegative θ-ratios, which are fine. Negative entries lead to nonpositive θ-ratios. In this case, there is no restriction imposed on how far the entering variable can be increased. For example, suppose the pivotal column in our example were

$$\begin{bmatrix} -2 \\ 5 \end{bmatrix} \quad \text{instead of} \quad \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Then we would have, instead of (12),

$$\begin{aligned} u &= \ \ 8 + 2x \\ v &= 15 - 5x. \end{aligned}$$

Since u must be nonnegative, we find that

$$8 + 2x \geq 0 \quad \text{or} \quad x \geq -4,$$

which puts no restriction on how far we can increase x. Thus, in calculating θ-ratios we can ignore any negative entries in the pivotal column.

If an entry in the pivotal column is zero, the corresponding θ-ratio is undefined. However, checking the equations corresponding to (12), but with one of the entries in the pivotal column equal to zero, will show that no restriction is placed on the size of x by the zero entry. Consequently, in forming the θ-ratios we use only the positive entries in the pivotal column that are above the objective row.

If all the entries in the pivotal column above the objective row are either zero or negative, then the entering variable can be made as large as we wish. Hence, the given problem has no finite optimal solution, and we can stop.

Forming a New Tableau

Having determined the entering and departing variables, we must obtain a new tableau showing the new basic variables and the new basic feasible solution. We illustrate the procedure with our continuing example. Solving the second equation of (8) (it corresponds to the departing variable) for x, the entering variable, we have

$$x = 3 - \tfrac{3}{5}y - \tfrac{1}{5}v. \tag{13}$$

Substituting (13) into the first equation of (8), we get

$$2(3 - \tfrac{3}{5}y - \tfrac{1}{5}v) + 2y + u = 8$$

or

$$\tfrac{4}{5}y + u - \tfrac{2}{5}v = 2. \qquad (14)$$

We also rewrite (13) as

$$x + \tfrac{3}{5}y + \tfrac{1}{5}v = 3. \qquad (15)$$

Substituting (13) into (7), we have

$$(-120)(3 - \tfrac{3}{5}y - \tfrac{1}{5}v) - 100y + z = 0$$

or

$$-28y + 24v + z = 360. \qquad (16)$$

Since in the new basic feasible solution we have $y = v = 0$, the value of z for this solution is 360. This value appears as the entry in the last row and rightmost column. Equations (14), (15), and (16) yield the new tableau (Tableau 2.3).

Tableau 2.3

	x	y	u	v	z	
u	0	$\frac{4}{5}$	1	$-\frac{2}{5}$	0	2
x	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	3
	0	-28	0	24	1	360

Observe that the basic variables in Tableau 2.3 are x and u. By comparing Tableaus 2.1 and 2.3, we see that the steps that were used to obtain Tableau 2.3 from Tableau 2.1 are as follows.

Step a. Locate and circle the entry at the intersection of the pivotal row and pivotal column. This entry is called the **pivot**. Mark the pivotal column by placing an arrow $\downarrow$ above the entering variable, and mark the pivotal row by placing an arrow $\leftarrow$ to the left of the departing variable.

Step b. If the pivot is k, multiply the pivotal row by $1/k$, making the entry in the pivot position in the new tableau equal to 1.

Step c. Add suitable multiples of the new pivotal row to all other rows (including the objective row), so that all other elements in the pivotal column become zero.

Step d. In the new tableau, replace the label on the pivotal row by the entering variable.

These four steps constitute a process called **pivoting**. Steps b and c use elementary row operations (see Section 0.2) and form one iteration of the procedure used to transform a given matrix to reduced row echelon form.

We now repeat Tableau 2.1 with the arrows placed next to the entering and departing variables and with the pivot circled (Tableau 2.1a).

Tableau 2.1a

		↓ x	y	u	v	z	
	u	2	2	1	0	0	8
←	v	(5)	3	0	1	0	15
		−120	−100	0	0	1	0

Tableau 2.3 was obtained from Tableau 2.1 by pivoting. We now repeat the process with Tableau 2.3. Since the most negative entry in the objective row of Tableau 2.3, −28, occurs in the second column, y is the entering variable of this tableau and the second column is the pivotal column. To find the departing variable we form the θ-ratios, that is, the ratios of the entries in the rightmost column (except for the objective row) to the corresponding entries of the pivotal column for those entries in the pivotal column that are positive. The θ-ratios are

$$\frac{2}{\frac{4}{5}} = \frac{5}{2} \quad \text{and} \quad \frac{3}{\frac{3}{5}} = 5.$$

The minimum of these is $\frac{5}{2}$, which occurs for the first row. Therefore, the pivotal row is the first row, the pivot is $\frac{4}{5}$, and the departing variable is u. We now show Tableau 2.3 with the pivot, entering, and departing variables marked (Tableau 2.3a).

Tableau 2.3a

		x	↓ y	u	v	z	
←	u	0	($\frac{4}{5}$)	1	$-\frac{2}{5}$	0	2
	x	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	3
		0	−28	0	24	1	360

We obtain Tableau 2.4 from Tableau 2.3 by pivoting. Since the objective row in Tableau 2.4 has no negative entries, we are finished, by the optimality criterion. That is, the indicated solution,

$$x = \tfrac{3}{2}, \quad y = \tfrac{5}{2}, \quad u = 0, \quad v = 0,$$

Tableau 2.4

	x	y	u	v	z	
y	0	1	$\frac{5}{4}$	$-\frac{1}{2}$	0	$\frac{5}{2}$
x	1	0	$-\frac{3}{4}$	$\frac{1}{2}$	0	$\frac{3}{2}$
	0	0	35	10	1	430

is optimal, and the maximum value of z is 430. Notice from Figure 1.14 that we moved from the extreme point $(0, 0)$ to the adjacent extreme point $(3, 0)$ and then to the adjacent extreme point $(\frac{3}{2}, \frac{5}{2})$. The value of the objective function started at 0, increased to 360, and then to 430, the entry in the last row and rightmost column.

Summary of the Simplex Method

We assume that the linear programming problem is in standard form and that $\mathbf{b} \geq \mathbf{0}$. In this case the initial basic feasible solution is

$$\mathbf{x} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix}.$$

In subsequent sections we will show how to extend the simplex method to other linear programming problems.

Step 1. Set up the initial tableau.

Step 2. Apply the optimality test: If the objective row has no negative entries in the labeled columns, then the indicated solution is optimal. Stop computation.

Step 3. Find the pivotal column by determining the column with the most negative entry in the objective row. If there are several possible pivotal columns, choose any one.

Step 4. Find the pivotal row. This is done by forming the **θ-ratios**—the ratios formed by dividing the entries of the rightmost column (except for the objective row) by the corresponding entries of the pivotal columns using only those entries in the pivotal column that are positive. The **pivotal row** is the row for which the minimum ratio occurs. If two or more θ-ratios are the same, choose one of the possible rows. If none of the entries in the pivotal column above the objective row is positive, the problem has no finite optimum. We stop our computation in this case.

Step 5. Obtain a new tableau by pivoting. Then return to Step 2.

In Figure 2.1 we give a flowchart and in Figure 2.2, a structure diagram for the simplex algorithm.

The reader can use the SMPX courseware described in Appendix C to experiment with different choices of pivot, observing how some choices

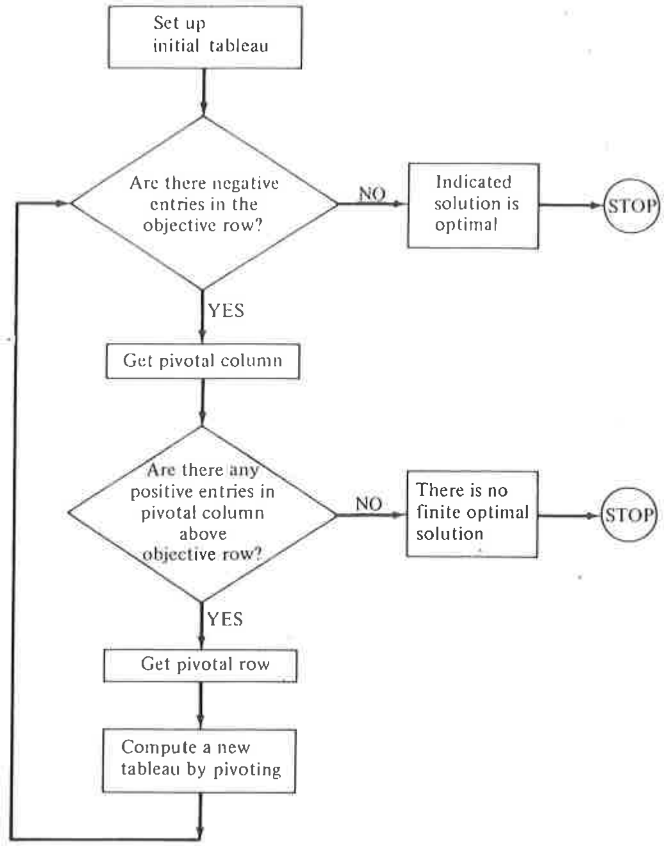

FIGURE 2.1 Flowchart for simplex algorithm (standard form, $\mathbf{b} \geq \mathbf{0}$).

lead to infeasible solutions. The courseware will also allow the user to step through the iterations of the simplex algorithm so that the intermediate tableaux can be examined.

The reader should note that the z column always appears in the form

z
0
0
⋮
0
1

in any simplex tableau. We included it initially to remind the reader that each row of a tableau including the objective row represents an equation

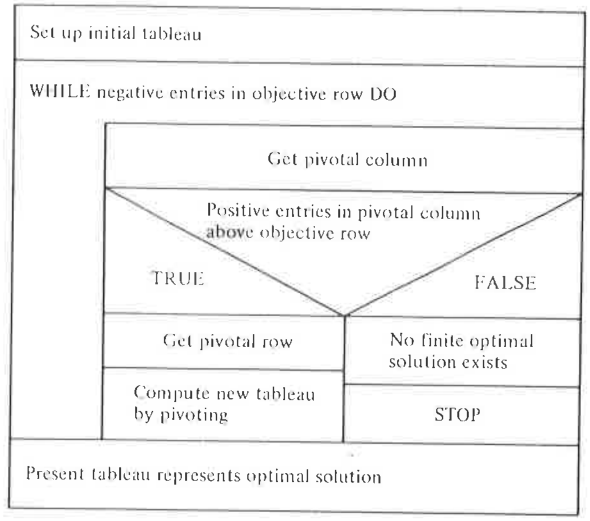

FIGURE 2.2 Structure diagram of simplex algorithm (standard form, $\mathbf{b} \geq \mathbf{0}$).

in the variables $x_1, x_2, \ldots, x_s, z$. From this point on we will not include the z column in tableaux. The student should remember to read the objective row of a tableau as an equation that involves z with coefficient $+1$.

EXAMPLE 2. We solve the following linear programming problem in standard form by using the simplex method:

$$\text{Maximize} \quad z = 8x_1 + 9x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 &\leq 2 \\ 2x_1 + 3x_2 + 4x_3 &\leq 3 \\ 6x_1 + 6x_2 + 2x_3 &\leq 8 \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0. \end{aligned}$$

We first convert the problem to canonical form by adding slack variables, obtaining:

$$\text{Maximize} \quad z = 8x_1 + 9x_2 + 5x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 \qquad\qquad &= 2 \\ 2x_1 + 3x_2 + 4x_3 \qquad + x_5 \qquad &= 3 \\ 6x_1 + 6x_2 + 2x_3 \qquad\qquad + x_6 &= 8 \\ x_j \geq 0, \quad j = 1, 2, \ldots, 6. & \end{aligned}$$

Tableau 2.5

		x_1	x_2 ↓	x_3	x_4	x_5	x_6	
	x_4	1	1	2	1	0	0	2
←	x_5	2	③	4	0	1	0	3
	x_6	6	6	2	0	0	1	8
		−8	−9	−5	0	0	0	0

Tableau 2.6

		x_1 ↓	x_2	x_3	x_4	x_5	x_6	
	x_4	$\frac{1}{3}$	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	1
	x_2	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1
←	x_6	②	0	−6	0	−2	1	2
		−2	0	7	0	3	0	9

Tableau 2.7

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{6}$	$\frac{2}{3}$
x_2	0	1	$\frac{10}{3}$	0	1	$-\frac{1}{3}$	$\frac{1}{3}$
x_1	1	0	−3	0	−1	$\frac{1}{2}$	1
	0	0	1	0	1	1	11

The initial tableau is Tableau 2.5; the succeeding tableaux are Tableaux 2.6 and 2.7.

Hence, an optimal solution to the standard form of the problem is

$$x_1 = 1, \quad x_2 = \tfrac{1}{3}, \quad x_3 = 0.$$

The values of the slack variables are

$$x_4 = \tfrac{2}{3}, \quad x_5 = 0, \quad x_6 = 0.$$

The optimal value of z is 11. △

EXAMPLE 3. Consider the linear programming problem

Maximize $z = 2x_1 + 3x_2 + x_3 + x_4$

subject to

$$\begin{aligned} x_1 - x_2 - x_3 \qquad\quad &\le 2 \\ -2x_1 + 5x_2 - 3x_3 - 3x_4 &\le 10 \\ 2x_1 - 5x_2 \qquad\quad + 3x_4 &\le 5 \\ x_j \ge 0, \quad j = 1,2,3,4. \end{aligned}$$

To solve this problem by the simplex method, we first convert the problem to canonical form by adding slack variables obtaining

$$\begin{aligned}
&\text{Maximize} \quad z = 2x_1 + 3x_2 + x_3 + x_4 \\
&\text{subject to} \\
&\quad x_1 - x_2 - x_3 + x_5 = 2 \\
&-2x_1 + 5x_2 - 3x_3 - 3x_4 + x_6 = 10 \\
&\quad 2x_1 - 5x_2 + 3x_4 + x_7 = 5 \\
&\quad x_j \geq 0, \quad j = 1, 2, \ldots, 7.
\end{aligned}$$

The initial tableau is Tableau 2.8; the following tableaux are Tableaux 2.9 and 2.10.

In Tableau 2.10, the most negative entry in the objective row is $-\frac{34}{3}$, so the departing variable is x_3. However, none of the entries in the pivotal column (the third column) is positive, so we conclude that the given problem has no finite optimal solution. △

Tableau 2.8

		x_1	x_2 ↓	x_3	x_4	x_5	x_6	x_7	
	x_5	1	−1	−1	0	1	0	0	2
←	x_6	−2	(5)	−3	−3	0	1	0	10
	x_7	2	−5	0	3	0	0	1	5
		−2	−3	−1	−1	0	0	0	0

Tableau 2.9

		x_1 ↓	x_2	x_3	x_4	x_5	x_6	x_7	
←	x_5	$\frac{3}{5}$	0	$-\frac{8}{5}$	$-\frac{3}{5}$	1	$\frac{1}{5}$	0	4
	x_2	$-\frac{2}{5}$	1	$-\frac{3}{5}$	$-\frac{3}{5}$	0	$\frac{1}{5}$	0	2
	x_7	0	0	−3	0	0	1	1	15
		$-\frac{16}{5}$	0	$-\frac{14}{5}$	$-\frac{14}{5}$	0	$\frac{3}{5}$	0	6

Tableau 2.10

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_1	1	0	$-\frac{8}{3}$	−1	$\frac{5}{3}$	$\frac{1}{3}$	0	$\frac{20}{3}$
x_2	0	1	$-\frac{5}{3}$	−1	$\frac{2}{3}$	$\frac{1}{3}$	0	$\frac{14}{3}$
x_7	0	0	−3	0	0	1	1	15
	0	0	$-\frac{34}{3}$	−6	$\frac{16}{3}$	$\frac{5}{3}$	0	$\frac{82}{3}$

2.1 EXERCISES

In Exercises 1 and 2, set up the initial simplex tableau.

1. Maximize $z = 2x + 5y$
   subject to

$$\begin{aligned} 3x + 5y &\le 8 \\ 2x + 7y &\le 12 \\ x \ge 0, \quad y &\ge 0. \end{aligned}$$

2. Maximize $z = x_1 + 3x_2 + 5x_3$
   subject to

$$\begin{aligned} 2x_1 - 5x_2 + x_3 &\le 3 \\ x_1 + 4x_2 \qquad &\le 5 \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 &\ge 0. \end{aligned}$$

3. Consider the following simplex tableau.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_4	0	0	2	1	$\frac{5}{2}$	0	0	$\frac{6}{7}$
x_1	1	0	5	0	-3	0	-2	$\frac{2}{7}$
x_6	0	0	3	0	4	1	-4	$\frac{5}{7}$
x_2	0	1	0	0	$\frac{3}{2}$	0	0	$\frac{1}{7}$

Determine the departing variable if the entering variable is (a) x_5; (b) x_3; (c) x_7.

In Exercises 4–7 use one iteration of the simplex algorithm to obtain the next tableau from the given tableau.

4.

	x_1	x_2	x_3	x_4	
x_4	$\frac{3}{2}$	0	$\frac{5}{3}$	1	6
x_2	$\frac{2}{3}$	1	2	0	8
	-4	0	-2	0	12

5.

	x_1	x_2	x_3	x_4	
x_1	1	2	0	1	3
x_3	0	$\frac{1}{2}$	1	-1	$\frac{3}{2}$
	0	-4	0	-4	$\frac{11}{2}$

6.

	x_1	x_2	x_3	x_4	x_5	
x_3	$\frac{2}{3}$	0	1	$\frac{3}{5}$	0	$\frac{3}{2}$
x_2	$\frac{3}{2}$	1	0	1	0	$\frac{5}{2}$
x_5	5	0	0	$\frac{2}{9}$	1	$\frac{2}{3}$
	4	0	0	-5	0	$\frac{7}{3}$

7.

	x_1	x_2	x_3	x_4	
x_2	1	1	5	0	4
x_4	-1	0	2	1	6
	-3	0	-2	0	7

8. (a) The following tableau arose in the course of using the simplex algorithm to solve a linear programming problem. What basic feasible solution does this tableau represent?

x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	$\frac{4}{3}$	$\frac{2}{3}$	0	1	0	$-\frac{1}{3}$	4
0	$\frac{1}{3}$	$\frac{2}{3}$	1	0	1	$-\frac{1}{3}$	10
1	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$	0	0	$\frac{1}{6}$	4
0	$-\frac{5}{3}$	$-\frac{4}{3}$	-1	0	0	$\frac{5}{3}$	12

(b) Perform one operation of the simplex algorithm on the tableau. What basic feasible solution does this new tableau represent?

9. Consider the following tableau, which arose in solving a linear programming problem by the simplex method.

x_1	x_2	x_3	u	v	w	
1	5	2	0	0	3	20
0	2	4	1	0	−4	6
0	2	−1	0	1	3	12
0	−5	−3	0	0	3	12

(a) Identify the basic feasible solution *and* basic variables in this tableau.
(b) Compute the next tableau using the simplex method.
(c) Identify the basic feasible solution *and* basic variables in the tableau in (b).

In Exercises 10–23 solve the indicated linear programming problem using the simplex method.

10. Example 4, Section 1.1.
11. Example 7a, Section 1.1.
12. Example 7b, Section 1.1.
13. Example 10, Section 1.1.
14. Exercise 4, Section 1.1.
15. Exercise 5, Section 1.1.
16. Exercise 7, Section 1.1.
17. Exercise 9, Section 1.1.
18. Exercise 2, Section 1.5.
19. Maximize $z = 2x_1 + 3x_2 - x_3$
subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &\le 6 \\ x_1 - 3x_2 - 3x_3 &\le 10 \\ x_j \ge 0, \quad j = 1, 2, 3. \end{aligned}$$

20. Maximize $z = x_1 + 2x_2 + x_3 + x_4$
subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &\le 8 \\ 2x_1 + 3x_2 + 4x_4 &\le 12 \\ 3x_1 + x_2 + 2x_3 &\le 18 \\ x_j \ge 0, \quad j = 1, 2, 3, 4. \end{aligned}$$

21. Maximize $z = 5x_1 + 2x_2 + x_3 + x_4$
subject to

$$\begin{aligned} 2x_1 + x_2 + x_3 + 2x_4 &\le 6 \\ 3x_1 + x_3 &\le 15 \\ 5x_1 + 4x_2 + x_4 &\le 24 \\ x_j \ge 0, \quad j = 1, 2, 3, 4. \end{aligned}$$

22. Maximize $z = -x_1 + 3x_2 + x_3$
    subject to

$$\begin{aligned} -x_1 + 2x_2 - 7x_3 &\leq 6 \\ x_1 + x_2 - 3x_3 &\leq 15 \\ x_j \geq 0, \quad j &= 1, 2, 3. \end{aligned}$$

23. Maximize $z = 3x_1 + 3x_2 - x_3 + x_4$
    subject to

$$\begin{aligned} 2x_1 - x_2 - x_3 + x_4 &\leq 2 \\ x_1 - x_2 + x_3 - x_4 &\leq 5 \\ 3x_1 + x_2 \qquad + 5x_4 &\leq 12 \\ x_j \geq 0, \quad j &= 1, 2, 3, 4. \end{aligned}$$

24. Suppose a linear programming problem has a constraint of the form

$$3x_1 + 2x_2 + 5x_3 - 2x_4 \geq 12.$$

Why can we not solve this problem using the simplex method as described up to this point? (In Section 2.3 we develop techniques for handling this situation.)

2.2 DEGENERACY AND CYCLING (OPTIONAL)

In choosing the departing variable, we computed the minimum θ-ratio. If the minimum θ-ratio occurs, say, in the rth row of a tableau, we drop the variable that labels that row. Now suppose that there is a tie for minimum θ-ratio, so that several variables are candidates for departing variable. We choose one of the candidates by using an arbitrary rule such as dropping the variable with the smallest subscript. However, there are potential difficulties any time such an arbitrary choice must be made. We now examine these difficulties.

Suppose that the θ-ratios for the rth and sth rows of a tableau are the same and their value is the minimum value of all the θ-ratios. These two rows of the tableau are shown in Tableau 2.11 with the label on the rth row marked as the departing variable. The θ-ratios of these two rows are

$$b_r / a_{rj} = b_s / a_{sj}.$$

Tableau 2.11

		x_1	x_2	$\cdots$	x_j ↓	$\cdots$	x_{n+m}	
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
←	x_{i_r}	a_{r1}	a_{r2}	$\cdots$	(a_{rj})	$\cdots$	$a_{r,n+m}$	b_r
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	x_{i_s}	a_{s1}	a_{s2}	$\cdots$	a_{sj}	$\cdots$	$a_{s,n+m}$	b_s
	$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$

Tableau 2.12

	x_1	x_2	$\cdots$	x_j	$\cdots$	x_{n+m}	
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_j	a_{r1}/a_{rj}	a_{r2}/a_{rj}	$\cdots$	1	$\cdots$	$a_{r,n+m}/a_{rj}$	b_r/a_{rj}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
x_{i_s}	*	*	$\cdots$	0	$\cdots$	*	$b_s - a_{sj} \cdot b_r/a_{rj}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$

When we pivot in Tableau 2.11, we obtain Tableau 2.12, where * indicates an entry whose value we are not concerned about. Setting the nonbasic variables in Tableau 2.12 equal to zero, we find that

$$x_j = b_r/a_{rj}$$

and

$$x_{i_s} = b_s - a_{sj} \cdot b_r/a_{rj} = a_{sj}(b_s/a_{sj} - b_r/a_{rj}) = 0.$$

Consequently, the tie among the θ-ratios has produced a basic variable whose value is 0.

DEFINITION. A basic feasible solution in which some basic variables are zero is called **degenerate**.

EXAMPLE 1 (DEGENERACY). Consider the linear programming problem in standard form

$$\text{Maximize} \quad z = 5x_1 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 &\le 2 \\ 2x_1 + x_2 &\le 4 \\ -3x_1 + 2x_2 &\le 6 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

The region of all feasible solutions is shown in Figure 2.3. The extreme points and corresponding values of the objective function are given in Table 2.1. The simplex method leads to the following tableaux. In Tableau 2.13 we have two candidates for the departing variable: x_3 and x_4 since the θ-ratios are equal. Choosing x_3 gives Tableaux 2.13, 2.14, 2.15, and 2.16. Choosing x_4 gives Tableaux 2.13a, 21.4a, and 2.15a. Note that Tableaux 2.15a and 2.16 are the same except for the order of the constraint rows.

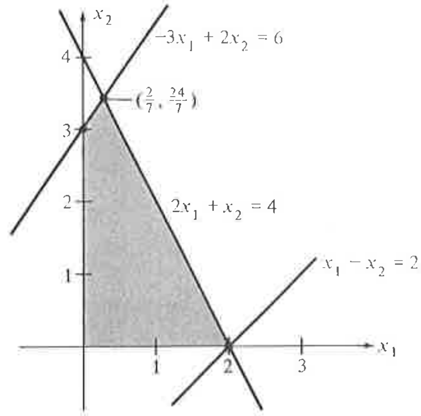

FIGURE 2.3

TABLE 2.1

Extreme point	*Value of* $z = 5x_1 + 3x_2$
(0, 0)	0
(2, 0)	10
(0, 3)	9
$(\frac{2}{7}, \frac{24}{7})$	$\frac{82}{7}$

Tableau 2.13

		↓ x_1	x_2	x_3	x_4	x_5	
←	x_3	①	−1	1	0	0	2
	x_4	2	1	0	1	0	4
	x_5	−3	2	0	0	1	6
		−5	−3	0	0	0	0

Tableau 2.14

		x_1	↓ x_2	x_3	x_4	x_5	
	x_1	1	−1	1	0	0	2
←	x_4	0	③	−2	1	0	0
	x_5	0	−1	3	0	1	12
		0	−8	5	0	0	10

Tableau 2.15

			↓			
		x_1	x_2	x_3	x_4	x_5	
	x_1	1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	2
	x_2	0	1	$-\frac{2}{3}$	$\frac{1}{3}$	0	0
←	x_5	0	0	($\frac{7}{3}$)	$\frac{1}{3}$	1	12
		0	0	$-\frac{1}{3}$	$\frac{8}{3}$	0	10

Tableau 2.16

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
x_2	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
x_3	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{36}{7}$
	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7}$

Tableau 2.13a

		↓					
		x_1	x_2	x_3	x_4	x_5	
	x_3	1	−1	1	0	0	2
←	x_4	(2)	1	0	1	0	4
	x_5	−3	2	0	0	1	6
		−5	−3	0	0	0	0

Tableau 2.14a

			↓				
		x_1	x_2	x_3	x_4	x_5	
	x_3	0	$-\frac{3}{2}$	1	$-\frac{1}{2}$	0	0
	x_1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	2
←	x_5	0	($\frac{7}{2}$)	0	$\frac{3}{2}$	1	12
		0	$-\frac{1}{2}$	0	$\frac{5}{2}$	0	10

Tableau 2.15a

	x_1	x_2	x_3	x_4	x_5	
x_3	0	0	1	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{36}{7}$
x_1	1	0	0	$\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
x_2	0	1	0	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{24}{7}$
	0	0	0	$\frac{19}{7}$	$\frac{1}{7}$	$\frac{82}{7}$

The optimal solution is

$$x_1 = \tfrac{2}{7}, \qquad x_2 = \tfrac{24}{7},$$

with the optimal value of the objective function being

$$z = \tfrac{82}{7}.$$

The slack variables have values

$$x_3 = \tfrac{36}{7}, \quad x_4 = 0, \quad x_5 = 0.$$

What is happening geometrically? We start with the initial basic feasible solution as the origin $(0, 0)$, where $z = 0$. If we choose to replace x_3 with x_1, we move to the adjacent extreme point $(2, 0)$, where $z = 10$ (Tableau 2.14). Now we replace x_4 with x_2 and remain at $(2, 0)$ (Tableau 2.15). Finally we replace x_5 with x_3 and move to $(\tfrac{2}{7}, \tfrac{24}{7})$, where $z = \tfrac{82}{7}$. This is our optimal solution (Tableau 2.16).

Alternatively, because the θ-ratios are equal we could replace x_4 with x_1. The pivot step with this choice again moves us from $(0, 0)$ to $(2, 0)$, where $z = 10$ (Tableau 2.14a). However, at the next stage, x_3, which has value 0 and is the degenerate variable, is not a departing variable. Instead, x_5 is the departing variable, and we move immediately to the optimal solution (Tableau 2.15a) at $(\tfrac{2}{7}, \tfrac{24}{7})$. △

In general, in the case of degeneracy, an extreme point is the intersection of too many hyperplanes. For example, degeneracy occurs in R^2 when three or more lines intersect at a point, degeneracy occurs in R^3 when four or more planes intersect at a point, and so on.

Cycling

If no degenerate solution occurs in the course of the simplex method, then the value of z increases as we go from one basic feasible solution to an adjacent basic feasible solution. Since the number of basic feasible solutions is finite, the simplex method eventually stops. However, if we have a degenerate basic feasible solution and if a basic variable whose value is zero departs, then the value of z does not change. To see this, observe that z increases by a multiple of the value in the rightmost column of the pivotal row. But this value is zero, so that z does not increase. Therefore, after several steps of the simplex method we may return to a basic feasible solution that we already have encountered. In this case the simplex method is said to be **cycling** and will never terminate by finding an optimal solution or concluding that no bounded optimal solution exists. Cycling can only occur in the presence of degeneracy, but many linear programming problems that are degenerate do not cycle (see Example 1).

Examples of problems that cycle are difficult to construct and rarely occur in practice. However, Kotiah and Steinberg (see Further Reading) have discovered a linear programming problem arising in the solution of a practical queuing model that does cycle. Also, Beale (see Further Reading) has constructed the following example of a smaller problem that cycles after a few steps.

EXAMPLE 2 (CYCLING). Consider the following linear programming problem in canonical form.

$$\text{Maximize} \quad z = 10x_1 - 57x_2 - 9x_3 - 24x_4$$

subject to

$$\begin{aligned} \tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 + x_5 \qquad\qquad &= 0 \\ \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 \qquad + x_6 \qquad &= 0 \\ x_1 \qquad\qquad\qquad\qquad\qquad\qquad + x_7 &= 1 \end{aligned}$$

$$x_j \geq 0, \quad j = 1, \ldots, 7.$$

Using the simplex method we obtain the following sequence of tableaux.

Tableau 2.17

		↓ x_1	x_2	x_3	x_4	x_5	x_6	x_7	
←	x_5	($\frac{1}{2}$)	$-\frac{11}{2}$	$-\frac{5}{2}$	9	1	0	0	0
	x_6	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	0	0
	x_7	1	0	0	0	0	0	1	1
		−10	57	9	24	0	0	0	0

Tableau 2.18

		x_1	↓ x_2	x_3	x_4	x_5	x_6	x_7	
	x_1	1	−11	−5	18	2	0	0	0
←	x_6	0	(4)	2	−8	−1	1	0	0
	x_7	0	11	5	−18	−2	0	1	1
		0	−53	−41	204	20	0	0	0

Tableau 2.19

		x_1	x_2	↓ x_3	x_4	x_5	x_6	x_7	
←	x_1	1	0	($\frac{1}{2}$)	−4	$-\frac{3}{4}$	$\frac{11}{4}$	0	0
	x_2	0	1	$\frac{1}{2}$	−2	$-\frac{1}{4}$	$\frac{1}{4}$	0	0
	x_7	0	0	$-\frac{1}{2}$	4	$\frac{3}{4}$	$-\frac{11}{4}$	1	1
		0	0	$-\frac{29}{2}$	98	$\frac{27}{4}$	$\frac{53}{4}$	0	0

Tableau 2.20

		x_1	x_2	x_3	↓ x_4	x_5	x_6	x_7	
	x_3	2	0	1	−8	$-\frac{3}{2}$	$\frac{11}{2}$	0	0
←	x_2	−1	1	0	⓶ (2)	$\frac{1}{2}$	$-\frac{5}{2}$	0	0
	x_7	1	0	0	0	0	0	1	1
		29	0	0	−18	−15	93	0	0

Tableau 2.21

		x_1	x_2	x_3	x_4	↓ x_5	x_6	x_7	
←	x_3	−2	4	1	0	($\frac{1}{2}$)	$-\frac{9}{2}$	0	0
	x_4	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	$\frac{1}{4}$	$-\frac{5}{4}$	0	0
	x_7	1	0	0	0	0	0	1	1
		20	9	0	0	$-\frac{21}{2}$	$\frac{141}{2}$	0	0

Tableau 2.22

		x_1	x_2	x_3	x_4	x_5	↓ x_6	x_7	
	x_5	−4	8	2	0	1	−9	0	0
←	x_4	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	(1)	0	0
	x_7	1	0	0	0	0	0	1	1
		−22	93	21	0	0	−24	0	0

Tableau 2.23

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	$\frac{1}{2}$	$-\frac{11}{2}$	$-\frac{5}{2}$	9	1	0	0	0
x_6	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	0	0
x_7	1	0	0	0	0	0	1	1
	−10	57	9	24	0	0	0	0

Observe that Tableau 2.23 is identical to Tableau 2.17, and, thus, the simplex method has cycled. △

Computer programs designed for large linear programming problems provide several options for dealing with degeneracy and cycling. One option is to ignore degeneracy and to assume that cycling will not occur. Another option is to use Bland's Rule for choosing entering and departing variables to avoid cycling. This rule modifies Step 3 and 4 of the Simplex Method.

Bland's Rule

1. **Selecting the pivotal column.** Choose the column with the smallest subscript from among those columns with negative entries in the objective row instead of choosing the column with the most negative entry in the objective row.

2. **Selecting the pivotal row.** If two or more rows have equal θ-ratios, choose the row labeled by the basic variable with the smallest subscript, instead of making an arbitrary choice.

Bland showed that if these rules are used, then, in the event of degeneracy, cycling will not occur and the simplex method will terminate.

EXAMPLE 3. Referring to the tableaux from Example 2, note that Bland's rule affects only the choice of entering variable in Tableau 2.22. Applying the rule and rewriting Tableau 2.22 with the new choice of entering and departing variables, we obtain Tableau 2.23a.

Tableau 2.22

		↓ x_1	x_2	x_3	x_4	x_5	x_6	x_7	
	x_5	-4	8	2	0	1	-9	0	0
←	x_4	($\frac{1}{2}$)	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	0	0
	x_7	1	0	0	0	0	0	1	1
		-22	93	21	0	0	-24	0	0

Tableau 2.23a

		x_1	x_2	↓ x_3	x_4	x_5	x_6	x_7	
	x_5	0	-4	-2	8	1	-1	0	0
	x_1	1	-3	-1	2	0	2	0	0
←	x_7	0	3	(1)	-2	0	-2	1	1
		0	27	-1	44	0	20	0	0

We perform the pivot step to obtain Tableau 2.24a, which represents an optimal solution. The cycling has been broken.

Tableau 2.24a

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	0	2	0	4	1	-5	2	2
x_1	1	0	0	0	0	0	1	1
x_3	0	3	1	-2	0	-2	1	1
	0	30	0	42	0	18	1	1

△

2.2 EXERCISES

In Exercises 1–6 solve the indicated linear programming problem, noting where degeneracies occur. Sketch the set of feasible solutions, indicating the order in which the extreme points are examined by the simplex algorithm.

1. Maximize $z = 6x_1 + 5x_2$
   subject to
$$\begin{aligned} 3x_1 - 2x_2 &\le 0 \\ 3x_1 + 2x_2 &\le 15 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

2. Maximize $z = 5x_1 + 4x_2$
   subject to
$$\begin{aligned} x_1 + 2x_2 &\le 8 \\ x_1 - 2x_2 &\le 4 \\ 3x_1 + 2x_2 &\le 12 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

3. Maximize $z = 3x_1 + 2x_2 + 5x_3$
   subject to
$$\begin{aligned} 2x_1 - x_2 + 4x_3 &\le 12 \\ 4x_1 + 3x_2 + 6x_3 &\le 18 \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 &\ge 0. \end{aligned}$$

4. Maximize $z = 5x_1 + 8x_2 + x_3$
   subject to
$$\begin{aligned} x_1 + x_2 + x_3 &\le 7 \\ 2x_1 + 3x_2 + 3x_3 &\le 12 \\ 3x_1 + 6x_2 + 5x_3 &\le 24 \\ x_1 \ge 0, \quad x_2 \ge 0, \quad x_3 &\ge 0. \end{aligned}$$

5. Maximize $z = 6x_1 + 5x_2$
   subject to
$$\begin{aligned} 4x_1 + 3x_2 &\le 19 \\ x_1 - x_2 &\le 3 \\ x_1 - 2x_2 &\le 2 \\ 3x_1 + 4x_2 &\le 18 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

6. Maximize $z = 5x_1 + 3x_2$
subject to

$$\begin{aligned} 2x_1 + x_2 &\le 6 \\ 2x_1 - x_2 &\ge 0 \\ x_1 - x_2 &\le 0 \\ x_j \ge 0, \quad j &= 1, 2. \end{aligned}$$

7. If a degeneracy arose in any of the exercises above, use all choices of the departing variable.

In Exercises 8 and 9,

(a) Show that cycling occurs when solving the problem using the simplex method.

(b) Use Brand's Rule to terminate cycling and obtain an optimal solution, if one exists.

8. Minimize $z = -x_1 + 7x_2 + x_3 + 2x_4$
subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ \tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 + x_6 &= 0 \\ \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 + x_7 &= 0 \\ x_j \ge 0, \quad j &= 1, \ldots, 7 \end{aligned}$$

(due to K. T. Marshall and J. W. Suurballe).

9. Minimize $z = -\tfrac{2}{5}x_1 - \tfrac{2}{5}x_2 + \tfrac{9}{5}x_3$
subject to

$$\begin{aligned} \tfrac{3}{5}x_1 - \tfrac{32}{5}x_2 + \tfrac{24}{5}x_3 + x_4 &= 0 \\ \tfrac{1}{5}x_1 - \tfrac{9}{5}x_2 + \tfrac{3}{5}x_3 + x_5 &= 0 \\ \tfrac{2}{5}x_1 - \tfrac{8}{5}x_2 + \tfrac{1}{5}x_3 + x_6 &= 0 \\ x_2 + x_7 &= 1 \\ x_j \ge 0, \quad j &= 1, \ldots, 7 \end{aligned}$$

(due to K. T. Marshall and J. W. Suurballe).

2.3 ARTIFICIAL VARIABLES

In the previous two sections we discussed the simplex algorithm as a method of solving certain types of linear programming problems. Recall that we restricted our discussion to those linear programming problems that could be put in standard form, and we considered only those problems in that form that had a *nonnegative right-hand side*. That is, we have