h-vectors and beyond

Bologna, May 2025

Abstract

"Proofs should only be communicated by consenting adults in private". (Victor Klee)

0 Definitions

Definition 1. A subset $A \subseteq \mathbb{R}^d$ is *convex* if for any two points $\mathbf{x}, \mathbf{y}$ of A, the entire segment

$$[\mathbf{x}, \mathbf{y}] \stackrel{\mathsf{def}}{=} \{t\mathbf{x} + (1-t)\mathbf{y} \,;\, 0 \le t \le 1\}$$

is contained in A. Given finitely many points $\mathbf{x_1}, \ldots, \mathbf{x_n}$ in A, a *convex combination* is a point $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{x_i}$, where $\sum_{i=1}^n \lambda_i = 1$ and $\lambda_i \ge 0$ for all i. The *convex hull of* A, denoted by $\operatorname{conv}(A)$, is the set of all convex combinations of points from A. In other words,

$$\operatorname{conv}(A) \stackrel{\mathsf{def}}{=} \{\sum_{i=1}^n \lambda_i \mathbf{a_i} \quad \text{such that } n \in \mathbb{N} \setminus \{0\}, \sum_{i=1}^n \lambda_i = 1,\ \lambda_i \ge 0 \text{ and } \mathbf{a_i} \in A \text{ for all } i\}.$$

Lemma 2. $\operatorname{conv}(A)$ *is the smallest convex subset containing* A.

Proof. Easy! Start by showing that $\operatorname{conv}(A)$ is convex... □

Definition 3. A *polytope* is the convex hull of finitely many points in $\mathbb{R}^d$.

Definition 4. Let P be a polytope in $\mathbb{R}^d$. A *face of* P is any subset $F \subset \mathbb{R}^d$ of the form

$$F = \{\mathbf{x} \in P \text{ such that } \mathbf{c} \cdot \mathbf{x} = c_0\},$$

where $\mathbf{c} \cdot \mathbf{x} \le c_0$ is an inequality satisfied by all $\mathbf{x}$ in P.
We also say that the linear inequality $\mathbf{c} \cdot \mathbf{x} \le c_0$ *supports the face* F *of* P.

In other words, "faces" are where linear functions are maximized within P. Faces may have different dimensions: For example, P is a face of itself, by taking $\mathbf{c} = \mathbf{0}$ and $c_0 = 0$. But also the empty set is a face of any polytope P, by taking $\mathbf{c} = \mathbf{0}$ and $c_0 = 1$.

Definition 5. Faces of dimension 0, 1 and $\dim P - 1$ are called *vertices*, *edges*, and *facets*.

Example 6 (Simplices)**.** We define the *standard d-dimensional simplex* as

$$\Delta^d \stackrel{\mathsf{def}}{=} \operatorname{conv}\{\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}\} \subseteq \mathbb{R}^{d+1}.$$

This can also be described in terms of facets as follows:

$$\Delta^d = \{\mathbf{x} \in \mathbb{R}^{d+1} \text{ such that } \mathbf{x}_i \ge 0 \text{ for all } i \text{ and } \sum_{i=1}^d \mathbf{x}_i = 1\}.$$

By construction, the d-simplex has $d+1$ vertices and $d+1$ facets. For example: Δ^1 is a segment of length $\sqrt{2}$ in $\mathbb{R}^2$; Δ^2 is an equilateral triangle in $\mathbb{R}^3$; Δ^3 is a regular tetrahedron in $\mathbb{R}^4$.

Example 7 (Cubes)**.** We define the *standard d-dimensional cube* as

$$C^d \overset{\text{def}}{=} \text{conv}\{\pm 1, \pm 1, \dots, \pm 1\} \subseteq \mathbb{R}^d.$$

This can also be described in terms of facets as follows:

$$C^d = \{\mathbf{x} \in \mathbb{R}^d \text{ such that } -1 \le \mathbf{x}_i \le +1 \text{ for all } i \}.$$

By construction, the d-cube has 2^d vertices and $2d$ facets. For example, C^1 is the segment $[-1, 1] \subseteq \mathbb{R}$, C^2 is a square, C^3 a cube.

Example 8 (Crosspolytopes)**.** We define the *standard crosspolytope* as

$$C^{d*} \overset{\text{def}}{=} \text{conv}\{\mathbf{e}_1, -\mathbf{e}_1, \dots, \mathbf{e}_d, -\mathbf{e}_d\} \subseteq \mathbb{R}^d.$$

This can also be described in terms of facets as follows:

$$C^{d*} = \{\mathbf{x} \in \mathbb{R}^d \text{ such that } \sum_{i=1}^{d} |\mathbf{x}_i| \le 1\}.$$

By construction, the cube has $2d$ vertices and 2^d facets. For example, C^{1*} is the segment $[-1, 1] \subseteq \mathbb{R}$, same as C^1; C^{2*} is the square C^2 rotated of 45 degrees; C^{3*} is the octahedron.

Definition 9 (Affine/Projective Equivalence)**.** Two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are *affinely equivalent* (resp. *projectively equivalent*) if there is an affine (resp. projective) map $f : \mathbb{R}^d \to \mathbb{R}^e$ that yields a bijection between P and Q when restricted to P.

Definition 10 (Combinatorial Equivalence)**.** Two polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$ are *combinatorially equivalent* if there is a bijection between the sets of their faces that preserves the inclusion relation. (Equivalently: if they have the same face poset.)

Proposition 11. *Affinely ⇒ projectively ⇒ combinatorially equivalent. Converses are false.*

Proof. The implications are obvious. Affine maps preserve parallelism, so a rectangle, a square and a parallelogram are all affinely equivalent, but a trapezoid is not affinely equivalent to them; however, all quadrilaterals are projectively equivalent. In general, any projective equivalence in d-space is determined by the image of $d + 2$ points; so if you perturb the position of just one of the vertices of a regular pentagon, say, you get a polygon with five edges no longer projectively equivalent to the regular one; but they would still be combinatorially equivalent. □

Typically, we have "combinatorial equivalence" in mind. So the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^d$ is called "a simplex".

Definition 12. The *graph* of a polytope is formed by all its faces of dimension ≤ 1. The *dual graph* of a d-dimensional polytope has nodes corresponding to its facets; two nodes are connected by an arc if and only if the corresponding facets are adjacent, i.e. if their intersection is $(d - 2)$-dimensional.

Remark 13. The graph does not determine the polytope, nor its dimension: for example, K_n is the graph of the $(n - 1)$-simplex, but also of infinitely many polytopes of dimension $d \ge 4$ (e.g. the "d-dimensional cyclic polytope on n vertices", for $n > d \ge 4$). Dual graphs determine *simplicial* polytopes, though (Blind–Mani–Kalai).

Remark 14. Most graphs are not graphs of polytopes. For example, a 2-edge path (and any graph containing it as induced subgraph) is not the graph of any polytope, for connectivity reasons. $K_{3,3}$ is 3-regular and 3-connected, but it is not the graph of any 3-dimensional polytope, because it is not planar. An even more interesting example, the graph obtained by removing a 7-cycle from K_8, is given by Grünbaum (pages 214–215 of his Convex Polytopes book).

1 Shellability

Definition 15. A *polytopal complex* is a finite, nonempty collection of polytopes, called "faces", closed under taking intersections and under taking faces (i.e. intersections with hyperplanes). *Simplicial complexes* (resp. *cubical complexes*) are polytopal complexes where all polytopes are simplices (resp. cubes). A polytope is called *simplicial* if its boundary is a simplicial complex. The inclusion-maximal polytopes in a polytopal complex, with slight abuse of notation, are called *facets*. A polytopal complex is *pure* if all its facets have same dimension.

Definition 16 (Shellability - simplicial case)**.** A pure simplicial complex C of dimension d with N facets is *shellable* if either $d(N-1)=0$, or one can order its facets $F_1, \ldots, F_N$ so that for each $j \geq 2$, the intersection of F_j with the union of the previous F_i's is pure $(d-1)$-dimensional.

Example 17. Shellable 1-dimensional complexes are just connected graphs.

Example 18. Let $d > 1$. The boundary $\partial\Delta^d$ of the d-simplex is 'extendably' shellable: any ordering of its facets is a shelling order. So any pure $(d-1)$-dimensional subcomplex of $\partial\Delta^d$ is shellable, and the shelling order can be continued to a shelling of $\partial\Delta^d$. (In contrast, the boundary of the octahedron is shellable, but not extendably.)

Because of Example 18, the next definition boils down to Definition 16 in the simplicial case:

Definition 19 (Shellability)**.** Let C be a pure polytopal d-complex with N facets.
A *shelling order for C* is:

- if $d = 0$, any ordering of the points of C;
- if $d \geq 1$, any order $F_1, \ldots, F_N$ of the facets of C so that for each $j \geq 2$, the intersection of F_j with the union of the previous F_i's is pure $(d-1)$-dimensional and admits a shelling order that can be extended to a shelling order of ∂F_j. (In particular, all the polytopes in C must have boundaries that admit shelling orders.)

A pure polytopal complex is called *shellable* if it admits a shelling order.

Theorem 20 (Bruggesser–Mani 1970)**.** *Let $d > 1$. The boundary of every d-polytope is shellable. In fact, for every boundary face F, there is a shelling of the boundary of the polytope in which all facets that contain F are listed before the others.*

Proof. "Rocket shelling". Place the polytope so that the origin is interior, and order the vertices of the dual polytope according to a generic linear function. The function can be chosen to have the barycenter of F as maximum (i.e. the rocket can lift off from the middle of F). □

In the previous theorem one can of course replace "before the others' with "after the others", because the reverse of a rocket shelling is also a rocket shelling. More generally:

Lemma 21 (Ziegler, Lemma 8.10)**.** *Let S be a shellable polytopal d-complex in which every $(d-1)$-face is in exactly two d-faces. If $F_1, F_2, \ldots, F_s$ is a shelling order for S, so is $F_s, F_{s-1}, \ldots, F_1$.*

Proof. For any F_j in the shelling, for any $(d-1)$-face G of F_j, there is a unique other facet F_i of S such that $G = F_i \cap F_j$. This other facet can appear either earlier $(i < j)$ or later $(i > j)$ than F_j. The roles are interchanged if we reverse the shelling of S, while also reversing (by induction on the dimension) the shellings of the boundaries of its facets. □

By induction, using Van Kampen's theorem or similar theorems in topology, one can actually say something about the topology:

Lemma 22 (Zeeman)**.** *Let S be a shellable polytopal d-complex in which every $(d-1)$-face is in at most two d-faces. Then S is homeomorphic to a ball or a sphere. More generally, if C is any shellable polytopal d-complex, then $\pi_1(C) = \ldots = \pi_{d-1}(C) = 0$.*

After Bruggesser–Mani's theorem, it is natural to ask whether every convex d-ball is shellable:

Example 23 (Rudin's ball)**.** If $F_1, \ldots, F_N$ is any shelling order for a d-ball, by Lemma 22 for any $j \leq N-1$ the complex $F_1 \cup \ldots \cup F_j$ is a shellable d-ball. But in 1958 Mary Ellen Rudin gave an example of a subdivision of a tetrahedron, with $\mathbf{f}$-vector $(14, 66, 94, 41)$, in which no minitetrahedron could possibly be the last in a shelling order, as the union of the other minitetrahedra is not a ball. Thus not all subdivisions of the tetrahedron are shellable.

There is a positive result on this though, if we are allowed to subdivide the complex further:

Theorem 24 (Adiprasito–Benedetti, 2017)**.** *If a polytonal d-complex C is convex, then the second barycentric subdivision of C is shellable.*

It is open whether already the first barycentric subdivision of any convex complex is shellable. Here is an example of the usefulness of shellability:

Theorem 25 (Euler's formula)**.** *For any d-polytope P,*

$$f_0 - f_1 + \ldots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d.$$

Proof. The *reduced Euler characteristic* of an arbitrary polytopal complex C is defined by

$$\chi(C) \stackrel{\text{def}}{=} -1 + f_0(C) - f_1(C) + \ldots + (-1)^{\dim C} f_{\dim C}.$$

It is easy to see that χ is additive, i.e. $\chi(C \cup D) + \chi(C \cap D) = \chi(C) + \chi(D)$. What we want to show is that $\chi(\partial P) = -(-1)^d$, or if you prefer, $\chi(\partial P) = (-1)^{d-1}$. Now given any d-polytope P, let $F_1, \ldots, F_N$ be a shelling order for ∂P. We claim that

$$\chi(F_1 \cup \ldots \cup F_j) = \begin{cases} 0, & \text{for } 1 \leq j \leq N-1 \\ (-1)^{d-1}, & \text{for } j = N. \end{cases}$$

Note that the claim implies our desired conclusion $\chi(\partial P) = (-1)^{d-1}$ (it's the $j = N$ case!), which in turns implies that $\chi(P) = (-1)^{d-1} + (-1)^d = 0$ for every d-polytope P. Let us show the claim by double induction, on the dimension d and on the number of facets j. When $d = 1$, P is a segment E with two endpoints, and our claim is clear: we have $j = N = 1$ and

$$\chi(E) = -1 + 2 - 1 = 0 = 1 - (-1)^0.$$

Now suppose $d \geq 2$. Since the F_i's are $(d-1)$-polytopes, by induction on the dimension we can assume that $\chi(F_i) = 0$. Moreover, the intersection of each F_j with the previous F_i is a shellable part of ∂F_j, but not the whole F_j unless $j = N$; thus again by induction on d we have that $\chi(F_j \cap \bigcup_{i<j} F_i) = 0$ if $j < N$ and $\chi(F_N \cap \bigcup_{i<N} F_i) = \chi(\partial F_i) = (-1)^{d-2}$. But then by induction on j, using additivity, we get $\chi(F_1 \cup \ldots \cup F_j) = 0$ if $j < N$; and again by additivity

$$\chi(F_1 \cup \ldots \cup F_N) = \chi(\bigcup_{i<N} F_i) + \chi(F_N) - \chi(F_N \cap \bigcup_{i<N} F_i) = 0 + 0 - (-1)^{d-2} = (-1)^{d-1}. \quad \square$$

Remark 26. Using tools that are not particularly difficult but are beyond the purpose of these lessons (like elementary knot theory) it is possible to show that in each dimension $d \geq 3$, there are many more shellable d-spheres than the boundaries of $(d+1)$-polytopes; and there are many more d-spheres than the shellable ones. In contrast, when $d = 2$ this gap disappears: There is a 100-year old theorem by Steinitz that says, "every polytopal complex homeomorphic to the 2-sphere is combinatorially equivalent to the boundary of some 3-polytope".

2 The h-vector

The face vector of (the boundary of) a d-dimensional polytope counts the number of faces in each dimension; since the empty face is by convention (-1)-dimensional, this results in a vector $(f_{-1}, f_0, \dots, f_{d-1})$ of d integers (so one more than the dimension of the complex!), with $f_{-1} = 1$. We wish to encode the f-vector of a $(d-1)$-complex in a polynomial, but "**backwards**". We'll be more precise soon; but the idea is that the *reverse* of $\mathbf{f}$ is easily obtained from the *reverse* of $\mathbf{h}$ by multiplying it by some invertible integer matrix U. (This U is the matrix that has $\binom{i}{j}$ in row i and column j, so it's upper triangular with 1s on the diagonal... which explains invertibility.)

Definition 27. Given a vector of integers $(f_{-1}, f_0, \dots, f_{d-1})$, with $f_{-1} = 1$, the f-polynomial is defined by

$$f(x) \stackrel{\text{def}}{=} f_{d-1} + f_{d-2}x + \ldots + f_{-1}x^d = \sum_{i=0}^{d} f_{i-1}x^{d-i}.$$

The h-polynomial is then defined by

$$h(x) = f(x-1) \qquad \text{or equivalently,} \qquad f(x) = h(x+1).$$

The coefficients of the h-polynomial are defined a posteriori, by the identity

$$h(x) = h_d + h_{d-1}x + h_{d-2}x + \ldots + h_1x^{d-1} + h_0x^d = \sum_{i=0}^{d} h_i x^{d-i}. \tag{1}$$

Remark 28. Notice three important details:
(1) The h-vector of a $(\mathbf{d-1})$-dimensional complex consists of $\mathbf{d+1}$ integers, $(h_0, \dots, h_d)$. We will see below that $h_0 = 1$ and $h_1 = f_0 - d$; moreover, the sum of the h_i's is just f_{d-1}.
(2) The definition above produces the h-vector *for any complex for which the f-vector is defined*; so in particular, for non-simplicial complexes. In fact, it makes sense even if f is a vector of arbitrary integers, like $(1, 5, 12)$ (which is not the f-vector of a graph – why?)
(3) Even if the f_i's are in $\mathbb{N}$, the h_i are in $\mathbb{Z}$ but not necessarily in $\mathbb{N}$, as the next example shows.

Example 29. Consider the 1-dimensional simplicial complex G consisting of the five edges 12, 13, 23, 24, 34. Note that both this facet order and its reverse are shelling orders. Now, G has f-vector $(1, 4, 5)$. As $f(x) \stackrel{\text{def}}{=} 5 + 4x + x^2$ and

$$h(x) \stackrel{\text{def}}{=} f(x-1) = 5 + 4(x-1) + (x-1)^2 = 2 + 2x + x^2,$$

the complex has h-vector $(1, 2, 2)$. Thus having a shelling order whose reverse is also a shelling order does not make the h-vector palindromic, as Ziegler seems to imply on pages 252.

Example 30. Consider the 2-dimensional simplicial complex C consisting of two triangles with a vertex in common. It has f-vector $(1, 5, 6, 2)$. As $f(x) \stackrel{\text{def}}{=} 2 + 6x + 5x^2 + x^3$ and

$$h(x) \stackrel{\text{def}}{=} f(x-1) = 2 + 6(x-1) + 5(x-1)^2 + (x-1)^3 = -x + 2x^2 + x^3,$$

the complex has h-vector $(1, 2, -1, 0)$. Note that one entry is negative.

Example 31. Consider the 2-dimensional polytopal complex D consisting of the boundary of the cube. It has f-vector $(1, 8, 12, 6)$. As $f(x) \stackrel{\text{def}}{=} 6 + 12x + 8x^2 + x^3$ and

$$h(x) \stackrel{\text{def}}{=} f(x-1) = 6 + 12(x-1) + 8(x-1)^2 + (x-1)^3 = 1 - x + 5x^2 + x^3,$$

this cubical complex has h-vector $(1, 5, -1, 1)$. Again, note that one entry is negative.

Theorem 32 (McMullen, Stanley)**.** *Let $(h_0, \dots, h_d)$ be the h-vector associated to an f-vector $(f_{-1}, f_0, \dots, f_{d-1})$.*

(I) $h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{d-k} f_{i-1}$, *so in particular* $h_0 = f_{-1} = 1$ *and* $h_1 = -df_{-1} + f_0 = f_0 - d$.

(II) $f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i}$, *so in particular* $f_{d-1} = h_0 + h_1 + \ldots + h_d$.

(III) *If we define the* reverse f-vector $\mathbf{f}^*$ *by* $f_i^* \stackrel{\text{def}}{=} f_{d-1-i}$ *for all* $i \in \{0, \dots, d\}$, *and the* reverse f-vector $\mathbf{h}^*$ *by* $h_i^* \stackrel{\text{def}}{=} h_{d-i}$, *then as matrices* $\mathbf{f}^* = U\mathbf{h}^*$, *where U is the upper triangular matrix defined by* $U_{i,j} \stackrel{\text{def}}{=} \binom{i}{j}$.

(IV) *There is a trick due to Stanley to compute h from f.*

(V) *In a* ***simplicial*** *complex that is the boundary of a d-dimensional (simplicial) polytope P, the h_i count the vertices of in-degree i in an orientation of the edges of the dual polytope P^* according to a generic linear functional. This implies $h_i \geq 0$ and $h_i = h_{d-i}$.*

(VI) *More generally, in a* ***simplicial*** *complex that is shellable and homeomorphic to the $(d-1)$-sphere, the h_i's count facets whose restriction set has size i, which still implies $h_i \geq 0$ and $h_i = h_{d-i}$.*

(VII) *More generally, in a* ***simplicial*** *complex that is homeomorphic to the $(d-1)$-sphere, we have no clue on what the h_i's count, but we can still prove that $h_i \geq 0$ and $h_i = h_{d-i}$.*

Proof. (I) By definition of h and f and by the Newton formula,

$$h(x) = f(x-1) = \sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} f_{i-i} \sum_{\ell=0}^{d-i} \binom{d-i}{\ell} x^{d-i-\ell}(-1)^{\ell}.$$

Let us compare the coefficients of x^{d-k} in the two sides of the polynomial identity above. On the left side, this is by definition h_k, cf. Equation (1). On the right hand side, once i is chosen, $d-i-\ell$ is equal to $d-k$ precisely when $\ell = k-i$; so we get

$$h_k = \sum_{i=0}^{d} f_{i-1} \binom{d-i}{k-i} (-1)^{k-i} = \sum_{i=0}^{d} f_{i-1} \binom{d-i}{d-k} (-1)^{k-i}.$$

But the binomial $\binom{d-i}{d-k}$ is 0 if $k > d$, so we may as well stop the sum at k.

(II) By definition of f and h and by the Newton formula,

$$f(x) = h(x+1) = \sum_{i=0}^{d} h_i(x+1)^{d-i} = \sum_{i=0}^{d} h_i \sum_{\ell=0}^{d-i} \binom{d-i}{\ell} x^{d-i-\ell}.$$

By equating the coefficient of x^{d-k} on both sides, we get

$$f_{k-1} = \sum_{i=0}^{d} h_i \binom{d-i}{k-i} = \sum_{i=0}^{d} h_i \binom{d-i}{d-k}, \tag{2}$$

a sum that can be stopped at $i = k$, since $\binom{d-i}{d-k} = 0$ for $k > d$.

(III) From Equation 2 above, for all $j \in \{0, \dots, d\}$ we get that

$$f_j^* \stackrel{\text{def}}{=} f_{(d-j)-1} \stackrel{!}{=} \sum_{i=0}^{d} h_i \binom{d-i}{d-j-i} = \sum_{i=0}^{d} h_i \binom{d-i}{j}.$$

Since $h_i^* \stackrel{\text{def}}{=} h_{d-i}$ for all i, via a reindexing trick we conclude

$$f_j^* = \sum_{i=0}^{d} h_i \binom{d-i}{j} = \sum_{\ell=0}^{d} h_{d-\ell} \binom{\ell}{j} = \sum_{i=0}^{d} h_{d-i} \binom{i}{j} = \sum_{i=0}^{d} h_i^* \binom{i}{j}.$$

(IV) Stanley's trick consists in

- placing $A_{0,0} \stackrel{\text{def}}{=} 1$ on top of an equilateral triangle with horizontal basis;
- writing elements $A_{0,\ell} \stackrel{\text{def}}{=} 1$ downwards on the left edge, until $\ell = d+1$.
- writing elements $A_{k,k} \stackrel{\text{def}}{=} f_{k-1}$, i.e. the f-vector downwards on the right edge.
- then computing $A_{k,\ell} \stackrel{\text{def}}{=} A_{k,\ell-1} - A_{k-1,\ell-1}$, i.e. each element is the difference of what's above on the right, and what's above on the left.

The trick is that $A_{k,d+1} = h_k$ for all $k \in \{0, \dots, d\}$; that is, "the h-vector appears horizontally on the bottom line", which is the $(d+2)$-nd. (Recall that the dimension of the complex was $d-1$, so when you stop you should see a table with three more rows than the dimension of the complex). Why does it work? If we define a map $a : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Z}$ by

$$a(k,\ell) \stackrel{\text{def}}{=} \sum_{i=0}^{k} (-1)^{k-i} \binom{\ell-1-i}{k-i} f_{i-1}$$

with the convention that $\binom{-n}{0} \stackrel{\text{def}}{=} 1$ for all $n \in \mathbb{N}$, it is easy to see that

- $a(0,\ell) = 1$,
- $a(k,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k-i-1}{k-i} f_{i-1} = f_{k-1}$,
- $a(k-1,k) = \sum_{i=0}^{k-1} (-1)^{k-1-i} f_{i-1} = f_{k-2} - f_{k-3} + \ldots + (-1)^k f_0 + (-1)^{k+1}$,
- and most importantly,

$$\begin{aligned} a(k,\ell-1) - a(k-1,\ell-1) &= \textstyle\sum_{i=0}^{k} (-1)^{k-i} \binom{\ell-2-i}{k-i} f_{i-1} - \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{\ell-2-i}{k-1-i} f_{i-1} = \\ &= \textstyle\sum_{i=0}^{k} (-1)^{k-i} \binom{\ell-2-i}{k-i} f_{i-1} + \sum_{i=0}^{k-1} (-1)^{k-i} \binom{\ell-2-i}{k-1-i} f_{i-1} = \\ &= \textstyle\sum_{i=0}^{k} (-1)^{k-i} f_{i-1} \left(\binom{\ell-2-i}{k-i} + \binom{\ell-2-i}{k-1-i} \right) = \\ &= \textstyle\sum_{i=0}^{k} (-1)^{k-i} f_{i-1} \binom{\ell-1-i}{k-i} \quad = a(k,\ell). \end{aligned}$$

Hence, $A_{k,\ell} = a(k,\ell)$ **for all** k, ℓ. And in particular

$$A_{k,d+1} = a(k,d+1) = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1} = h_k.$$

Note that Stanley's trick works for numerical reasons, so it would also work if we wanted to compute the h-vector of a non-simplicial complex from its f-vector, say.

(V) The poset of the faces of P^* is the poset of the faces of P "upside down"; in particular, $f_r(P^*) = f_{d-1-r}(P)$. Since P is simplicial, the graph of P^*, which is the dual graph of P, is d-regular, and every i edges incident to some vertex $\mathbf{x}$ of P^* uniquely determine an i-face of P^* that includes them. Now, fix a drawing $\mathfrak{D}$ of P^* in $\mathbb{R}^d$ so that all its vertices have different quotas. Orient all the edges of P^* downwards. This induces an orientation on the graph of P^* with the following obvious properties:

(a) the orientation is acyclic;
(b) each face has exactly one sink and one source (the highest resp. lowest vertex);
(c) any vertex has in-degree k if and only if it has out-degree $d-k$;
(d) any vertex of in-degree k will be a sink in exactly 2^k faces (because as we said above any choice of i of these k edges uniquely determines one such face).

Let $H_r(P^*, D)$ count the vertices of in-degree r with respect to the drawing $\mathfrak{D}$. This defines a vector $\mathbf{H}(P^*, \mathfrak{D}) = (H_0(P^*, \mathfrak{D}), \ldots, H_d(P^*, \mathfrak{D}))$ of non-negative integers adding up to $f_0(P^*)$. Note that with an "upside-down drawing" $\mathfrak{D}'$ that reverses the orientation of all edges of the graph of P^*, vertices that had in-degree r become of in-degree $d-r$

with the drawing $\mathfrak{D}'$; so $H_{d-r}(P^*, \mathfrak{D}') = H_r(P^*, \mathfrak{D})$. Next, we claim that for every drawing $\mathfrak{D}$ chosen,

$$f_k(P^*) = \sum_{r=0}^{d} H_r(P^*, \mathfrak{D}) \binom{r}{k}. \tag{3}$$

Let us prove the claim. Let v be a vertex of P^* that has in-degree r in the chosen drawing. The number of k-faces containing v and for which v is "sink" is exactly $\binom{r}{k}$, because any k edges incident to v span a k-face. Since each k-face has exactly one sink, the right-hand side above counts all k-faces exactly once. So Equation (3) is proven. Equation (3) has two immediate yet powerful consequences:

- the vector $H = H(P^*, \mathfrak{D})$ depends only on P^*, because Equation (3) tells us that, up to an invertible constant matrix U, H equals to the f-vector of P^*. In particular, $H_{d-r}(P^*, \mathfrak{D}) = H_{d-r}(P^*, \mathfrak{D}') = H_r(P^*, \mathfrak{D})$. So the vector H is palindromic!
- If we compare part (II) of the present theorem with the equation

$$f_{k-1}(P) = f_{d-k}(P^*) = \sum_{r=0}^{d} H_{d-r}(P^*, \mathfrak{D}) \binom{r}{d-k} = \sum_{i=0}^{d} H_i(P^*, \mathfrak{D}) \binom{d-i}{k-i} \tag{4}$$

  we see that $H_i(P^*, \mathfrak{D}) = h_i(P)$ for all i. In other words, H is the h-vector of P!

(VI) Let C be a shellable simplicial complex of dimension $d-1$. Let $F_1, \ldots, F_N$ be a shelling order for its facets. For any $j \in \{1, \ldots, N\}$, define the "restriction set"

$$R_j \stackrel{\text{def}}{=} \{v \in F_j \text{ such that } \operatorname{del}(v, F_j) \subseteq F_i \text{ for some } i < j\}.$$

In other words, R_j is the face spanned by the vertices v of F_j with the property that the deletion of v from F_j is already contained in one of the earlier facets.

We claim that in a shelling order, the "new faces" added in the j-th step (i.e. the faces contained in $F_1 \cup \ldots \cup F_{j-1} \cup F_j$, but not in $F_1 \cup \ldots \cup F_{j-1}$) are exactly the faces G such that $R_j \subseteq G \subseteq F_j$. Let us show the claim: let $G \subseteq F_j$ be a face not contained in $F_1 \cup \ldots \cup F_{j-1}$. By contradiction, suppose that some vertex v of R_j is not in G. Then G belongs to the deletion of v. So by definition of R_j, G is contained in some face F_i with $i < j$, a contradiction. So all vertices of R_j are in G, that is, $R_j \subseteq G$. Conversely, let G be a face such that $R_j \subseteq G \subseteq F_j$. Being in F_j, clearly $G \subseteq F_1 \cup \ldots \cup F_{j-1} \cup F_j$. By contradiction, suppose that G is contained in $F_1 \cup \ldots \cup F_{j-1}$, or in other words, that G is in also some F_h with $h < j$. Then $G \subseteq F_h \cap F_j$, which means that $\dim(F_h \cap F_j) = \dim F_j - 1$. Let w be the vertex in F_j not in $F_j \cap F_h$. Since $R_j \subseteq (F_h \cap F_j) = \operatorname{del}(w, F_j)$, we have $w \notin R_j$. At the same time, since $\operatorname{del}(w, F_j) \subseteq F_h$, we would have $w \in R_j$; a contradiction. So the claim is proven. But then by induction on j, if $F_1, \ldots, F_N$ is a shelling order, then C is the disjoint union of the "intervals" $[R_1, F_1]$, $\ldots$, $[R_N, F_N]$. In particular all the $(k-1)$-faces of C are partitioned into intervals $[R_j, F_j]$. Each one of these intervals, if $|R_j| = i$, contains exactly $\binom{d-i}{k-i}$ faces of dimension $k-1$ (because the F_j are all simplices, so the poset $[R_j, F_j]$ is Boolean, i.e. it's isomorphic to the poset of all subsets of a given finite set.) So for each i in $\{0, \ldots, d\}$ set

$$r_i(C) \stackrel{\text{def}}{=} |\{j \in \{1, \ldots, s\} \text{ such that } |R_j| = i\}|.$$

Since there are exactly $r_i(C)$ intervals $[R_j, F_j]$ with $|R_j| = i$, since these intervals are disjoint, and since each one of them contains exactly $\binom{d-i}{k-i}$ faces of dimension $k-1$,

$$f_{k-1}(C) = \sum_{i=0}^{k} r_i(C) \binom{d-i}{k-i}. \tag{5}$$

Comparing this with item (II) above tells us that

$$h_i = r_i(C).$$

Thus $h_i \geq 0$. So far we never used the "sphere" assumption. We do need it to show that $h_k = h_{d-k}$ though: Compare Example 29. Details are not so easy, but the bottom line is: In a sphere, not only the reverse of a shelling on a sphere is again a shelling (as we saw in Lemma 21), but also, the restriction set for F_j in the reverse shelling is exactly $\operatorname{del}(R_j, F_j)$, the complement of the restriction for the shelling we started with. Since $|R_j| = k$ if and only if $|\operatorname{del}(R_j, F_j)| = d - k$, we conclude.

(VII) This part is too difficult to explain quickly, but it is proven via commutative algebra: To every simplicial complex C with n vertices one can bijectively associate a radical monomial ideal I_C in $S \overset{\text{def}}{=} \mathbb{R}[x_1, \ldots, x_n]$. The quotient of S by I_C is called *Stanley-Reisner ring of C*. If we start with a simplicial complex C homeomorphic to a sphere, this ring turns out to be Cohen–Macaulay (which is the reason for $h_i \geq 0$) and even Gorenstein (which is the reason for $h_k = h_{d-k}$). □

The proof of part (VI) of Theorem 32 actually proves a few more facts.

Theorem 33 (Seidel)**.** *Any (simplicial) shellable $(d-1)$-sphere on n vertices has a number of facets bounded above by a polynomial in n of degree $\lfloor d/2 \rfloor$.*

Proof. We give a proof for simplicial spheres, but we can reduce ourselves to the simplicial case by subdividing all facets into simplices "without adding extra vertices". For $d = 2$ the claim is obvious: a polygon with n vertices has exactly n edges. For $d = 3$, from Euler's formula $n - e + f = 2$ and from $3r = 2e$ (which comes from the simplicial assumption) we get that the number f of facets is exactly $2n - 4$. For $d \geq 4$, the number of vertices no longer determines the number of facets. However, set $\ell \overset{\text{def}}{=} \lfloor d/2 \rfloor \geq 2$. Fix a shelling. For any facet F_j, let R_j be its restriction set. Clearly, either R_j has size at most ℓ, or its complement $\operatorname{del}(R_j, F_j)$ has size at most ℓ. So either in our shelling or in its reverse, the facet F_j has a restriction set of size $\leq \ell$. Since the association "facet $\to$ restriction set" is injective, we get

$$f_{d-1} \leq 2 \cdot |\{\text{k-faces with } k \leq \ell\}| \leq 2 \sum_{i=0}^{\ell} \binom{n}{i}.$$ □

Definition 34. A d-dimensional simplicial complex P is called *partitionable* if for each facet F_j there is a face R_j (called "*the restriction set of F_j*" such that for every face F of P, there is exactly one $j \in \{1, \ldots, N\}$ such that $R_j \subseteq F \subseteq F_j$. (In other words, a simplicial complex is partitionable if its face poset can be partitioned into intervals that all stop at a facet.)

Proposition 35. *Every shellable simplicial complex is partitionable.*

Proposition 36. *Every partitionable simplicial complex has $h_i \geq 0$ for all i, since the $h_i(P)$ counts the facets of P whose restriction set has size i.*

Definition 37. A simplicial complex with facets $F_1, \ldots, F_N$ is *doubly-partitionable* if there are faces $R_1, \ldots, R_N$ such that:

(i) for every face F of P, there is exactly one $i \in \{1, \ldots, N\}$ such that $R_i \subseteq F \subseteq F_i$, and
(ii) for every face F of P, there is exactly one $j \in \{1, \ldots, N\}$ such that $\operatorname{del}(R_j, F) \subseteq F \subseteq F_j$.

Proposition 38. *Every shellable sphere is doubly-partitionable.*

Proposition 39. *Every $(d-1)$-dimensional doubly-partitionable simplicial complex has h-vector satisfying $h_i = h_{d-i}$.*

3 Some recent developments

This section collects together a few results (typically without proof) about f-vectors. The first result is a conjecture by Imre Baranyi, recently proven by Joshua Hinman:

Proposition 40 (Hinman 2022)**.** *Let $0 \leq k < d$ be integers. Let P be any d-polytope. Then*

$$\frac{f_k(P)}{f_0(P)} \geq \frac{1}{2}\left[\binom{\lceil d/2 \rceil}{k} + \binom{\lfloor d/2 \rfloor}{k}\right] \qquad \text{and} \qquad \frac{f_k(P)}{f_{d-1}(P)} \geq \frac{1}{2}\left[\binom{\lceil d/2 \rceil}{d-k-1} + \binom{\lfloor d/2 \rfloor}{d-k-1}\right].$$

In particular, $f_k(P) \geq \min(f_0(P), f_{d-1}(P))$.

It is natural to conjecture (Motzkin) the unimodality of the f-vectors of polytopes. This is however false, even in the simplicial case:

Proposition 41 (Eckhoff, Björner)**.** *The f-vectors of simplicial d-polytopes are unimodal if and only if $d \leq 19$: There is a simplicial 20-polytope with $f_{11} > f_{12} < f_{13}$.*

Proposition 42 (Björner 1981)**.** *Let $d \geq 3$. For simplicial d-polytopes*

$$1 = f_{-1} < f_0 < \ldots < f_{\lfloor d/2 \rfloor - 1} \leq f_{\lfloor d/2 \rfloor} \qquad \text{and } f_{\lfloor 3(d-1)/4 \rfloor} > \ldots > f_{d-2} > f_{d-1},$$

and this is best possible: for any p, d such that $\lfloor d/2 \rfloor \leq p \leq \lfloor 3(d-1)/4 \rfloor$, there is an f-vector whose maximum entry is f_p.

Note that there is no contradiction between the two statements: The second one implies that all f-vectors of polytopes of dimension 10 or less, are unimodal. In fact, for $d = 19$, the second statement tells you that $1 < f_0 < \ldots < f_8 \leq f_9 \leq f_{10}$ and $f_{13} > \ldots > f_{17} > f_{18}$.

A *log-concave* sequence is a sequence that satisfies $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. It is easy to see that if a finite sequence (a_i) is non-negative, log-concave, and without internal zeroes, then it is unimodal.

Proposition 43 (Major 2013)**.** *The f-vectors of neighborly polytopes are log-concave.*

Neighborly polytopes on n vertices are those with graph K_n. It took quite long to establish the previous result, even if we did have a formula counting the number of k-faces of any neighborly polytopes:

Proposition 44. *Let $0 \leq k \leq d$ be integers. Let P be any neighborly d-polytope on $f_0 = n > d$ vertices. If d is odd,*

$$f_{k-1} = 2 \sum_{i=0}^{\frac{d-1}{2}} \binom{n-d-1+i}{i}.$$

If d is even,

$$f_{k-1} = \binom{n-d-1+d/2}{d/2} + 2 \sum_{i=0}^{\frac{d-2}{2}} \binom{n-d-1+i}{i}.$$

Proof. See Ziegler, pages 255–257. □

Upper Bound Theorem

The following theorem was conjectured by McMullen (first part) and Klee (second part):

Theorem 45 (UBT, McMullen 1970, Stanley 1975)**.** *Among all d-polytopes with n vertices, the one with the most k-faces, for any k, is any neighborly polyope. In fact, the previous statement remains true if we replace "among all d-polytopes" with "among all polytopal $(d-1)$-complexes homeomorphic to the $(d-1)$-sphere".*

Proof sketch. For polytopes, McMullen's inductive proof is sketched in Ziegler's book. Proving the UBT for polytopes is equivalent to proving it for simplicial polytopes, because from a polytope P we can always obtain a simplicial polytope P' with same vertices, but more faces. This can be shown with a convexity argument (namely, a slight perturbation of the position of the vertices in non-simplicial facets) or with a purely combinatorial "pulling triangulation" (as done by Richard Stanley).

It remains to show the UBT for $(d-1)$-spheres. Via the Stanley-Reisner correspondence, simplicial complexes correspond to certain squarefree monomial ideals. Since $f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i}$, the problem of maximizing f_{k-1} is equivalent to the problem of maximizing the h_i's. These h_i's are dimensions of certain vector spaces, studied by Hilbert and Macaulay, and they satisfy certain inequalities described by Macaulay. More specifically, if Δ is a $(d-1)$-dimensional Cohen–Macaulay complex on n vertices, if we call $\mathbb{K}[\Delta] \stackrel{\text{def}}{=} \mathbb{K}[x_1, \ldots, x_n] / I_\Delta$, then there exists a "regular sequence" $\theta_1, \ldots, \theta_d$ of d elements of $\mathbb{K}[\Delta]$, and we have

$$\mathrm{Hilb}\left(\mathbb{K}[\Delta] / (\theta_1, \ldots, \theta_d) \right) = (1-t)^d \,\mathrm{Hilb}\left(\mathbb{K}[\Delta]\right) = (1-t)^d \sum_{i=0}^{d} \frac{h_i(\Delta) t^i}{(1-t)^d} = \sum_{i=0}^{d} h_i(\Delta) t^i.$$

But since $\mathbb{K}[\Delta] / (\theta_1, \ldots, \theta_d)$ is generated as a $\mathbb{K}$-algebra by $n-d$ elements of degree one, the h_i cannot exceed the number of monomials of degree i in $n-d$ variables. From this we get an upper bound for the h_i's of the form

$$h_i \leq \binom{n-d-1+i}{i},$$

with equality for all k if and only if the complex is "neighborly". □

Lower bound theorem

Every d-polytope has at least $d+1$ vertices, so $1 \leq -d + f_0$, which can be rewritten as

$$h_0 \leq h_1.$$

Grünbaum first noticed the following fact:

- the f-vector of 2-polytopes satisfies $f_1 = f_0$;
- the f-vector of simplicial 3-polytopes satisfies $f_1 = 3f_0 - 6$;
- the f-vector of simplicial 4-polytopes satisfies $f_1 \geq 4f_0 - 10$;
- the f-vector of simplicial 5-polytopes satisfies $f_1 \geq 5f_0 - 15$.

Thus he conjectured that for *simplicial* d-polytopes, $d \geq 3$, $f_1 \geq df_0 - \binom{d+1}{2}$. This was proven in 1970 by Barnette, who noticed that for $d \geq 2$ such equation can be equivalently rewritten as $-d + f_0 \leq \frac{d(d-1)}{2} - (d-1)f_0 + f_1$, or in other words,

$$h_1 \leq h_2.$$

Note that for simplicial 2-polytopes, $h_1 \leq h_2$ is false, since $h_1 = -2 + f_0$ and $h_2 = 1$. But this is expected, because of the Dehn–Sommerville equations: if $h_0 \leq h_1$, we are going to have $h_d \leq h_{d-1}$. This triggered the question of whether perhaps the h**-vector** of polytopes is unimodal, even if the f-vector isn't. This turns out to be true!

Definition 46 (Murai–Nevo, 2012)**.** A triangulation of a d-manifold with boundary M is i-*stacked* if it has no interior faces of dimension $\leq d - i - 1$. (Or, equivalently, if all faces of dimension $d - i - 1$ are on the boundary.)

Theorem 47 (Generalized Lower Bound Theorem, Barnette 1973, Stanley 1975, Murai–Nevo 2012)**.** *For every simplicial (homology)* $(d-1)$*-sphere* S*,*

$$h_0 \leq h_1 \leq h_2 \leq \ldots \leq h_m, \qquad \text{where } m = \lfloor \frac{d}{2} \rfloor.$$

Moreover, if there exists a triangulated d*-manifold* M *with boundary equal to* S*, then* $h_i(S) = h_{i+1}(S)$ *for some* $i < \frac{d-1}{2}$ *if and only if* M *is* i*-stacked.*

Remark 48. Adiprasito–Benedetti very recently showed that a PL manifold is i-stacked if and only if it admits a handle decomposition into handles of index $\leq i$. This way it is possible to construct infinitely many simplicial homology spheres of dimension 6 that are not homeomorphic, and all have $h_2 = h_3$.

Remark 49. The GLBT holds for simplicial complexes only. For arbitrary polytopes, we don't even have a good conjecture (and also, the h_i's behave differently: We no longer have Dehn–Sommerville or non-negativity). Recently Lei Xue proved a conjecture by Grünbaum for polytopes with $d + s$ vertices, with $0 \leq s \leq 2d$: the number of k-faces is at least

$$\binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.$$

Remark 50. The h-vectors of a simplicial polytope is not necessarily log-concave. This is because it can be shown that if the h-vector of a complex is log-concave, so is its f-vector. Yet Björner's example from Proposition 41 has an f-vector that is not log-concave.

The g-theorem

The fact that the h is palindromic and weakly-increasing half the way suggests to define a new vector that encodes the same information as h (or f):

Definition 51. The g*-vector* is defined by

$$g_0 \stackrel{\text{def}}{=} 1 \qquad \text{and} \qquad g_k = h_k - h_{k-1} \text{ for } 1 \leq k \leq \lfloor d/2 \rfloor.$$

The lower bound theorem is equivalent to $\mathbf{g} \geq 0$. So can we get an upper bound for $\mathbf{g}$ that implies the upper bound theorem? The answer is positive, but a bit technical.

Definition 52. For any positive integers k, n, there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i}$$

such that $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Define

$$\partial_k(n) \stackrel{\text{def}}{=} \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_i}{i-1} \text{ and}$$

$$\partial^k(n) \stackrel{\text{def}}{=} \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \ldots + \binom{a_i - 1}{i-1}.$$

Example 53. Set $k = 3$, $n = 7$. Then since $7 = \binom{4}{3} + \binom{3}{2}$, we have

$$\partial^3(7) = \binom{3}{2} + \binom{2}{1} = 5 \qquad \text{and} \qquad \partial_3(7) = \binom{4}{2} + \binom{3}{1} = 9.$$

Definition 54. A sequence $(a_0, \ldots, a_d)$ is called
- a *K-sequence* if $a_{k-1} \geq \delta_k(a_k)$ for all k in $\{1, \ldots, d\}$;
- an *M-sequence* if $a_0 = 1$ and $a_{k-1} \geq \delta^k(a_k)$ for all k in $\{1, \ldots, d\}$.

Theorem 55 (Macaulay 1927; Kruskal 1963, Katona 1968, Clements–Lindström 1969)**.** *Let $d \geq 1$. Let $a_0, \ldots, a_d$ be integers.*
(1) *$(a_0, \ldots, a_d)$ is a K-sequence $\Longleftrightarrow$ the vector $(1, a_0, \ldots, a_d)$ is the f-vector of a d-dimensional simplicial complex.*
(2) *$(a_0, \ldots, a_d)$ is an M-sequence $\Longleftrightarrow$ there is a finitely generated graded $\mathbb{K}$-algebra R over some field $\mathbb{K}$, such that $R_0 = \mathbb{K}$, R_1 generates R, and $\dim_{\mathbb{K}} R_i = a_i$ for all i.*

Theorem 56 (g-theorem; Stanley 1979, Billera–Lee 1979, Adiprasito 2018, Karu–Xiao 2022)**.** *Let $d \geq 2$. Let $g_0, \ldots, g_{\lfloor d/2 \rfloor}$ be integers. Let $\mathbf{g} \stackrel{\text{def}}{=} (g_0, \ldots, g_{\lfloor d/2 \rfloor})$. The following are equivalent:*
(a) $\mathbf{g}$ *is an M-sequence;*
(b) $\mathbf{g}$ *is the g-vector of a simplicial d-polytope;*
(c) $\mathbf{g}$ *is the g-vector of a $(d-1)$-sphere;*
(d) $\mathbf{g}$ *is the g-vector of a $(d-1)$-dimensional homology-sphere.*

Here (a) implies (b) is due to Billera–Lee; (b) implies (a), to Stanley; (d) implies (a), to Adiprasito. Recently Karu–Xiao have shared a simpler proof that (c) implies (a).

3.1 Open Problems

(1) (Ziegler) Can one characterize the f-vectors of (not necessarily simplicial) polytopes?
(2) (Kalai) If G is a planar graph, then $f_1 \leq 3f_0$. Is it true that every 2-complex C that embeds in $\mathbb{R}^4$ satisfies $f_2(C) \leq 4 \cdot f_0(C)$? Is there a function $A : \mathbb{N} \to \mathbb{N}$ such that every d-complex C that embeds in $\mathbb{R}^{2d}$ satisfies $f_d(C) \leq A(d) \cdot f_0(C)$?
(3) (3^d-conjecture, Kalai) Is it true that centrally symmetric d-polytope has $\geq 3^d$ faces? (Stanley proved it for the simple/simplicial case.)
(4) (Cube-simplex) For every positive integer k, is there a d such that every polytope of dimension $\geq d$ has either a k-face that is a simplex, or a k-face that is a cube?
(5) (Fatness) The *fatness* of a 4-polytope is $\frac{f_1+f_2}{f_0+f_3}$. Can it be arbitrarily large?
(6) (Dürer) Does every convex polytope have a non-overlapping edge unfolding?
(7) (Partitionability) The disjoint union of K_2 and K_3 is a 1-dimensional complex that is partitionable, but not shellable. Is there a partitionable non-shellable sphere?