CONVEXITY OF COVERINGS OF PROJECTIVE VARIETIES AND VANISHING THEOREMS

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Abstract. In this article we propose properties of the algebra of holomorphic functions on universal covers of projective varieties. In particular, the following weakened version of the Shafarevich conjecture is proposed: The universal covering $\tilde{X}$ of a projective variety X, $\rho : \tilde{X} \to X$, is holomorphically convex modulo the pre-image, $\rho^{-1}(Z)$, of a subvariety $Z \subset X$. We prove this conjecture for projective varieties X having a negative vector bundle V having a cocycle $s \in H^1(X, V)$ whose pullback $\rho^* s$ is a coboundary. Our method of the proof also gives a new argument to obtain vanishing results of the first cohomology for negative vector bundles V over X whose rank is smaller than the dimension of X.

1. Introduction

In the early 70's I. Shafarevich proposed the following conjecture: The universal cover of a projective variety is holomorphically convex. This conjecture has been proved in some cases (see [Ka95]and [EKPR] for the strongest results), but the general case has remained unreachable. The lack of methods to construct holomorphic functions on universal covers and the possible existence of infinite chains of compact subvarieties in the universal cover of a projective manifold have been the main obstructions to proving this conjecture. An infinite chain of compact analytic subvarieties of the universal covering is the main geometrical obstruction to the universal covering being holomorphic convex. The first author and L. Katzarkov produced some examples of algebraic surfaces that possibly contain infinite chains [BoKa98].

In this paper we propose a weakened version of the Shafarevich conjecture and also other related questions/conjectures concerning the abundance of holomorphic functions on universal covers of projective varieties. The weakened version of the Shafarevich conjecture consists of: the universal cover $\tilde{X}$ of a projective variety X, $\rho : \tilde{X} \to X$, is holomorphic convex of modulo of an analytic subvariety $\rho^{-1}(Z)$, with Z a subvariety of X (i.e for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z, there exists a holomorphic function f on $\tilde{X}$ which unbounded

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on the sequence). The interest of this conjecture is that it is still very strong but does not exclude the existence of infinite chains of compact subvarieties. The strength of our weakened conjecture is manifested in the fact that it would still separate universal covers of projective varieties from universal covers of compact non-kahler manifolds with many holomorphic functions. The example to have in mind is the case of the universal cover of an Hopf surface which is $\mathbb{C}^2 \setminus \{(0,0)\}$, this complex manifold has many holomorphic functions but it is not holomorphic convex modulo of the pre-image of a any subvariety of the Hopf surface.

The conjectures proposed in this paper are motivated by a new method to produce holomorphic functions on the universal covers of projective varieties. There are two main aspects of the method. The first deals with the holomorphic properties of affine bundles associated with nontrivial extensions of semi-negative vector bundles on a projective variety X. The second deals with the trivialization of nontrivial extensions of semi-negative vector bundles on a projective variety X, when they are pulled back to the universal cover $\tilde{X}$. Our method to produce holomorphic functions on the universal coverings of projective varieties is different from the classical method of using the pullback of holomorphic differential (1,0)-forms on the projective variety X to construct holomorphic functions on the universal cover $\tilde{X}$.

An interesting application of our method is a new simple proof of the vanishing of the first cohomology of negative vector bundles whose rank is smaller than the dimension of the base.

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2. Background and preliminaries

2.1 Function theory and holomorphic convexity of universal covers.

We recall and propose some properties of the algebra of holomorphic functions $\mathcal{O}(X)$ on a complex manifold X. Since we are specially interested in universal covers of projective manifolds some of these properties are tailored to this case.

A complex manifold X is *holomorphic convex* if for every infinite discrete sequence $\{x_i\}_{i\in\mathbb{N}}$ of points in X there exists a holomorphic function f on X which unbounded on the sequence. Shafarevich proposed the following:

Conjecture. *(Shafarevich) The universal cover of a projective variety is holomorphic convex.*

It is unclear what was the evidence that Shafarevich used to support his conjecture. It was known that holomorphic convexity is a property that is shared by the compact and noncompact universal covers of Riemann surfaces. It was also known that if the fundamental group of a Kahler variety X is abelian then its universal cover $\tilde{X}$ is holomorphic convex. But maybe, the most inspiring evidence was the result of Poincare [Si73] stating that if a bounded domain D in an complex Euclidean space is the universal cover of a projective manifold then D is holomorphic convex.

The Shafarevich Conjecture predicts that noncompact universal covers of projective manifolds have many holomorphic functions. The holomorphic convexity of the universal cover implies that there is a proper map into $\mathbb{C}^n$. In particular, holomorphic convexity implies that there are enough holomorphic functions to separate points that are not connected by a connected chain of compact analytic subvarieties. This pointwise holomorphic separability property is the strongest possible for a complex manifold.

We propose a weakened version of holomorphic convexity, that will appear in section 3 to generalize the Shafarevich conjecture.

Definition 2.1. *Let X be a complex manifold and $\rho : \tilde{X} \to X$ the universal covering of X. The universal cover X is holomorphic convex modulo an analytic subset $\rho^{-1}(Z)$, $Z \subset X$, if for every infinite discrete sequence $\{x_i\}_{i\in\mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z, there exists a holomorphic function f on $\tilde{X}$ which unbounded on the sequence.*

Both holomorphic convexity and holomorphic convexity modulo an analytic subset of a universal cover imply the abundance of holomorphic functions on $\tilde{X}$. We are interested in other measurements of this abundance. An important geometric property of the algebra of holomorphic functions on a complex manifold X is given by studying the maps of X into complex Euclidean spaces.

Definition. *The holomorphic dimension of a complex manifold X, $\dim_h X$, is the largest k such that it exists an holomorphic map $f : X \to \mathbb{C}^k$ which is a submersion at some $x \in X$.*

There is also a dimension that one can associate with a projective or Kahler manifold X, that clearly influences the function theory of its universal cover. In work of Kollar [Ko93] and Campana [Ca 94] on the Shafarevich conjecture, it was shown that every projective (Kahler) manifold X has a dominant connected rational (meromorphic) map to a normal variety (analytic space) $Sh(X)$, $sh : X \dashrightarrow Sh(X)$ such that:

- there are countably many closed proper subvarieties $D_i \subset X$ such that for every irreducible $Z \subset X$ with $Z \not\subset \bigcup D_i$, one has: Z is contracted by the map sh if and only if $\mathrm{im}[\pi_1(Z) \to \pi_1(X)]$ is finite.

The map $sh : X \dashrightarrow Sh(X)$ is called the *Shafarevich map* and $Sh(X)$ is called the *Shafarevich variety of X*.

Definition. *The Shafarevich dimension of a projective (Kahler) manifold X is $\dim_{Sh} X = \dim_{\mathbb{C}} Sh(X)$.*

An immediate relation between these two dimensions is:

Proposition 2.1. *If X is a projective (Kahler) manifold then $\dim_h \tilde{X} \leq \dim_{Sh} X$.*

Proof. Let $U \subset X$ and $V \subset Sh(X)$ be Zariski open such that $sh|_U : U \to V$ is a morphism. The pre-image under $\rho : \tilde{X} \to X$ of any fiber of $sh|_U$ is a union of compact analytic subvarieties. Therefore through the general point $x \in \tilde{X}$ passes a compact analytic subvariety C_x of dimension $\dim_{\mathbb{C}} X - \dim_{Sh} X$ (C_x is maximal, i.e not contained in another compact analytic subvariety). The result follows since the C_x's must be in the fibers of any $f : \tilde{X} \to \mathbb{C}^k$. ☐

There is a generalization of the notion of holomorphic convexity. Let X be a complex manifold and L a line bundle on X with an Hermitean metric h. X is *holomorphic convex with respect to* (L, h) if for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in X there exists a section $s \in H^0(X, L)$ such that the function $|s|_h$ is unbounded on the sequence (holomorphic convexity is the special case of holomorphic convexity with respect to the trivial line bundle equipped with the trivial metric). It is important for us to recall a result from [Na90] concerning holomorphic convexity with respect to positive line bundles. Let X be a smooth projective variety and L a positive line bundle on X then there exists a $p \gg 0$ such that the universal cover $\tilde{X}$, $\rho : \tilde{X} \to X$, is holomorphic convex with respect to $(\rho^* L^p, h)$, h is any continuous Hermitean metric on L^p.

2.2 Affine bundles and the negativity of vector bundles.

Let X be a complex manifold and V a vector bundle of rank r on X. We will use the common abuse of notation that denotes by V also the sheaf of sections of V.

There is a 1-1 natural correspondence between cocycles $s \in H^1(X, V)$ and extensions of V by $\mathcal{O}$:

$$0 \to V \to V_s \to \mathcal{O} \to 0 \tag{2.1}$$

The trivial extension is given by the trivial cocycle. An affine bundle can be associated the extension (2.1). The preimage in V_s of a nonzero constant section m of the trivial line bundle is an affine bundle. This affine bundle is independent of the choice of m and is denoted by A_s. The affine bundle A_s is a vector bundle if and only if (2.1) splits or equivalently if s is cohomologous to zero. Also recall that an affine bundle is a vector bundle if and only if it has a section.

The affine bundle A_s can be described in an alternative way. Fix the following notation, let E be a vector bundle of rank r over X then $p : \mathbb{P}(E) \to X$ is the $\mathbb{P}^{r-1}$-bundle over X, whose points in the fiber $\mathbb{P}(E_x)$ are the hyperplanes in the vector space E_x, $x \in X$. Associated to a surjection $E \to F \to 0$ one has an inclusion $\mathbb{P}(F) \subset \mathbb{P}(E)$. The affine bundle A_s is $\mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$, where the inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V_s^*)$ comes from the (2.1) dualized.

A line bundle L on X is $k-ample$ if there is an N such that for $n \geq N$ the morphism $i_{|L^n|} : X \to \mathbb{P}^l$ has at most k-dimensional fibers. A 0-ample line bundle is the same as an ample line bundle.

Let V be a vector bundle over a complex manifold X. We recall characterizations of negativity (semi-negativity) for vector bundles. The projective bundle $\mathbb{P}(V)$ has a naturally defined line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on it, given by $\mathcal{O}_{\mathbb{P}(V)}(1) = p^*E/F$, F the tautological hyperplane bundle over $\mathbb{P}(V)$. A vector bundle is $k-ample$ if the line $\mathcal{O}_{\mathbb{P}(V)}(1)$ is k-ample. A vector bundle V is $k-negative$ if $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ is k-ample. We call V negative (in the sense of Grauert) if it is $0-negative$.

A complex manifold X is *q-convex* in the sense of Andreotti-Grauert if there is a C^∞ function $\varphi : X \to \mathbb{R}$ such that outside a compact subset $K \subset X$:

(i) The subset $\{x \in X : \varphi(x) < c \text{ with } c < \sup_X \varphi\}$ is relative compact.

(ii) $\varphi|_{X \setminus K}$ is q-convex i.e. the Levi form $L(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_i} dz_i \otimes d\bar{z}_j$ has at most $q-1$ non-positive eigenvalues at all $x \in X \setminus K$

The vector bundle V being negative is equivalent to the total space, $t(V)$, of V being 1-convex (strongly pseudoconvex). To recall relevant facts about projective bundles, we show how negativity of V implies that $t(V)$ is strongly pseudoconvex. Consider the surjection $V^* \oplus \mathcal{O} \to V^*$, its induced inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V^* \oplus \mathcal{O})$ and the expression $N_{\mathbb{P}(V^*)/\mathbb{P}(V^* \oplus \mathcal{O})} = \mathcal{O}_{\mathbb{P}(V^*)}(1)$. The positivity of the normal bundle $N_{\mathbb{P}(V^*)/\mathbb{P}(V^* \oplus \mathcal{O})}$ implies that complement of $\mathbb{P}(V^*)$ in $\mathbb{P}(V^* \oplus \mathcal{O})$:

$$\mathbb{P}(V^* \oplus \mathcal{O}) \setminus \mathbb{P}(V^*) \simeq V \tag{2.2}$$

is strongly pseudoconvex.

A generalization of the previous argument gives that A_s is strongly pseudoconvex if V is negative. Recall that for any surjection $E \to F$ and respective inclusion $\mathbb{P}(F) \subset \mathbb{P}(E)$ the expression for the normal bundle is:

$$N_{\mathbb{P}(F)/\mathbb{P}(E)} \simeq p^*(E^*/F^*) \otimes \mathcal{O}_{\mathbb{P}(F)}(1) \tag{2.3}$$

where $p : \mathbb{P}(F) \to X$ is the projection.

The isomorphism (2.3) gives:

$$N_{\mathbb{P}(V^*)/\mathbb{P}(V_s^*)} = \mathcal{O}_{\mathbb{P}(V^*)}(1) \tag{2.4}$$

and hence if V is negative the affine bundle A_s is strictly pseudoconvex.

We are interested in vector bundles V over compact complex manifolds such that the total space $t(V)$ has nonconstant holomorphic functions. The negativity or k-negativity of a vector bundle V implies that $t(V)$ has many nonconstant holomorphic functions. The following result illustrates a very strong geometric property of the algebra of holomorphic functions on $t(V)$ for a k-negative vector bundle V.

Proposition 2.2. *If V is a k-negative vector bundle over a complex manifold X then there is a proper holomorphic map $r : t(V) \to \mathbb{C}^k$ such that excluding the zero section $Z(V)$ the fibers of r are at most k-dimensional.*

Proof. Below, we construct a proper map $f : t(\mathcal{O}_{\mathbb{P}(V^*)}(-1)) \to \mathbb{C}^{l+1}$ such that excluding the zero section $Z(\mathcal{O}_{\mathbb{P}(V^*)}(-1))$ the fibers of r are at most k-dimensional. Assuming the existence of f, we construct the map $r : t(V) \to \mathbb{C}^{l+1}$. The blow up of $t(V)$ along the zero section $Z(V)$ is $Bl_{Z(V)}t(V) \equiv t(\mathcal{O}_{\mathbb{P}(E^*)}(-1))$, denote the blow up map by $\sigma : t(\mathcal{O}_{\mathbb{P}(E^*)}(-1)) \to t(V)$. The map f descends via σ to the desired map r and since σ is a biholomorphism away from the zero sections the map r has the desired properties.

The k-negativity of V implies that there is a n and a basis for $H^0(\mathbb{P}(E^*), \mathcal{O}_{\mathbb{P}(E^*)}(n))$ $\{s_0, ..., s_l\}$ such that the map $i_{|\mathcal{O}_{\mathbb{P}(V^*)}(n)|} : \mathbb{P}(V^*) \to \mathbb{P}^l$ given by $x \to [s_0(x), ..., s_l(x)]$ is a morphism with at most k-dimensional fibers. On the other hand a section $s \in H^0(\mathbb{P}(V^*), \mathcal{O}_{\mathbb{P}(V^*)}(n))$ gives a holomorphic function f_s on $t(\mathcal{O}_{\mathbb{P}(V^*)}(-1))$ that is homogeneous of order n on each fiber. The map $f = (f_{s_0}, ..., f_{s_l}) : t(\mathcal{O}_{\mathbb{P}(V^*)}(-1)) \to \mathbb{C}^{l+1}$ is our desired map.

The composition of the restriction of f to $t(\mathcal{O}_{\mathbb{P}(V^*)}(-1)) \setminus Z(\mathcal{O}_{\mathbb{P}(V^*)}(-1))$ with the projection $p : \mathbb{C}^{l+1} \backslash 0 \to \mathbb{P}^l$ descends to $\mathbb{P}(E^*)$ and the resulting map is exactly $i_{|\mathcal{O}_{\mathbb{P}(V^*)}(n)|}$. The map f is proper since the line bundle $\mathcal{O}_{\mathbb{P}(V^*)}(n))$ is globally generated. All fibers of F that are not $Z(\mathcal{O}_{\mathbb{P}(V^*)}(-1))$ descend to fibers of $i_{|\mathcal{O}_{\mathbb{P}(V^*)}(n)|}$ and hence are at most k-dimensional. $\square$

The following is a method to construct many negative bundles of rank $\geq \dim X$ with nontrivial first cohomology. One possible construction goes as follows. Let L be a very ample line bundle on X which gives an embedding $X \subset \mathbb{P}^n$. There is a surjective map $h : \mathcal{O}_X^{\oplus n+1} \to L$ which defines a rank n subbundle $\ker h = F \subset \mathcal{O}_X^{\oplus n+1}$. The extension

$$0 \to F \otimes L^{-1} \to \bigoplus^{n+1} L^{-1} \to \mathcal{O} \to 0 \tag{2.5}$$

is the pullback of the Euler exact sequence of $\mathbb{P}^n$ to X. The vector bundle $F \otimes L^{-1}$ is a negative bundle, $F \otimes L^{-1} \cong \Omega^1_{\mathbb{P}^n|X}$, and $H^1(X, F \otimes L^{-1}) \neq 0$. Namely there is a nontrivial element $s \in H^1(X, F \otimes L^{-1}) \neq 0$ which corresponds to the above nontrivial extension.

The following standard lemma is also important to keep in mind.

Lemma 2.1. *Let $f : Y \to X$ a flat finite morphism between projective varieties X and Y and V a vector bundle over X. If $s \in H^i(X, V)$ is nontrivial then $f^*s \in H^i(Y, f^*V)$ is also nontrivial.*

Proof. The map f being finite implies that $H^i(Y, f^*V)$ is canonically isomorphic to $H^i(X, f_*f^*V)$. The result follows since the natural map $\mathcal{O}_X \to f_*\mathcal{O}_Y$ splits using the trace $Tr_{Y/X} : f_*\mathcal{O}_Y \to \mathcal{O}_X$ and therefore V is a direct summand of f_*f^*V. $\square$

3. Convexity properties of universal covers

The universal cover $\tilde{X}$ of a projective variety X is Stein if and only if the nontrivial extension of the negative vector bundle $\Omega^1_{\mathbb{P}^n|X}$ coming from the Euler sequence becomes the trivial extension when it is pulled back to $\tilde{X}$ (see corollary 3.2). Let us consider the more general case of a projective variety X having a negative or a semi-negative vector bundle V with a nontrivial extension by $\mathcal{O}$ that becomes the trivial extension when it is pulled back to the universal cover $\tilde{X}$. We will reveal some properties of the algebra of holomorphic functions on $\tilde{X}$.

Theorem 3.1 below shows that if a projective variety X and a negative vector bundle V are as above then the universal cover $\tilde{X}$ is holomorphic convex modulo $\rho^{-1}(Z)$, Z a subvariety of X. The fact that $\tilde{X}$ is not quite holomorphic convex is an important characteristic of the result, since it allows us to slightly weaken the Shafarevich conjecture and at the same time allow infinite chains of compact subvarieties on the universal covers (the main geometric obstruction to the conjecture). Theorem 3.1 suggests the following weakened Shafarevich conjecture:

Conjecture. *The universal covering $\tilde{X}$ of a projective variety X is holomorphically convex modulo the pre-image of a subvariety $Z \subset X$.*

3.1 Negative vector bundles.

We start this section by proving the conjecture if a projective variety satisfies the conditions described above for a negative vector bundle.

Theorem 3.1. *Let X be a projective variety with a negative vector bundle V and $\rho : \tilde{X} \to X$ its universal covering. If it exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^* s = 0$ then $\tilde{X}$ is holomorphic convex modulo $\rho^{-1}(Z)$, Z is a subvariety of X.*

Proof. First, we will identify the subvariety $Z \subset X$ described in the theorem. As described in the section 2.2, the nontrivial cocycle $s \in H^1(X, V)$ has associated with it the strongly pseudoconvex affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ originating from the nonsplit exact sequence:

$$0 \to \mathcal{O} \to V_s^* \to V^* \to 0 \tag{3.1}$$

The strongly pseudoconvex manifold A_s (hence holomorphic convex) has a proper holomorphic map onto a Stein space, $r : A_s \to St(A_s)$ (the Remmert reduction). Moreover, A_s has a subset M called the maximal compact analytic subset of A_s such that the map $r|_{A_s \setminus M} : A_s \setminus M \to r(A_s \setminus M)$, is a biholomorphism. The subvariety $Z \subset X$ is $Z = p(M)$.

Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points in $\tilde{X} \setminus \rho^{-1}(Z)$ such that $\{\rho(x_i)\}_{i \in \mathbb{N}}$ has no accumulation points on Z. The sequence $\{x_i\}_{i \in \mathbb{N}}$ has a subsequence $\{y_i\}_{i \in \mathbb{N}}$ satisfying

$\{\rho(y_i)\}_{i\in\mathbb{N}}$ converges to $a \in X \setminus Z$. It is enough to show that $\exists\ f \in \mathcal{O}(\tilde{X})$ that is unbounded on the subsequence $\{y_i\}_{i\in\mathbb{N}}$.

Let L be a positive line bundle on X, Napier's result [Na90] states that for $p \gg 0$ $\exists$ $s \in H^0(\tilde{X}, \rho^*L^p)$ such that $|s(x_n)|_{\rho^*h^p}$ is unbounded (h is an Hermitean metric on L). Let $s' \in H^0(X, L^p)$ be such that $a \notin D = (s')_0$. The meromorphic function $h = \frac{s}{\rho^* s'}$ is holomorphic outside $\rho^{-1}(D)$ and unbounded on $\{y_i\}_{i\in\mathbb{N}}$. Assume the existence of a $q \in \mathcal{O}(\tilde{X})$ vanishing on $\rho^{-1}(D)$ and not vanishing at $\rho^{-1}(a)$. Then for l sufficiently large $f = hq^l$ would be the desired holomorphic function.

Claim: There $\exists q \in \mathcal{O}(\tilde{X})$ vanishing on $\rho^{-1}(D)$ and not vanishing at $\rho^{-1}(a)$.

To obtain holomorphic functions on $\tilde{X}$ we will construct a holomorphic map $g : \tilde{X} \to A_s$ such that $g(\tilde{X}) \not\subset M$ and pullback the holomorphic functions of A_s to $\tilde{X}$. The pullback of the exact sequence (3.1) to the universal covering $\tilde{X}$ splits into:

$$0 \to \mathcal{O} \to \mathcal{O} \oplus \rho^*V^* \to \rho^*V^* \to 0$$

since it is associated with the trivial cocycle $\rho^*s \in H^1(\tilde{X}, \rho^*V)$. As observed in (2.2) $A_{\rho^*s} \equiv \mathbb{P}(\rho^*V_s^*) \setminus \mathbb{P}(\rho^*V^*) \simeq \rho^*V$, hence $\tilde{X}$ embeddeds in A_{ρ^*s} as the zero section of ρ^*V^*. The affine bundle A_{ρ^*s} is the fiber product $A_{\rho_s^*} = \tilde{X} \times_X A_s$, denote the projection to the second factor by $\rho' : A_{\rho^*s} \to A_s$ and the embedding of $\tilde{X}$ in A_{ρ^*s} as the zero section of the vector bundle ρ^*V^* by $s : \tilde{X} \to A_{\rho^*s}$. The holomorphic map $g : \tilde{X} \to A_s$ will be the composition $g = \rho' \circ s : \tilde{X} \to A_s$. The map g is a local biholomorphism between $\tilde{X}$ and $g(\tilde{X})$ hence the condition $g(\tilde{X}) \not\subset M$ will hold if $\dim M < dimX$.

The maximal compact analytic subset of A_s $M = \cup_{i=1}^k M_i$, M_i the compact irreducible positive dimensional subvarieties of A_s, is such that $\dim M_i < dimX$ as the following proposition shows.

Proposition 3.1. *Let X be a projective variety with a vector bundle V and V_s be the extension associated with a nontrivial cocycle $s \in H^1(X, V)$. Then any compact subvariety M of the affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ satisfies* $\mathrm{dim} M < \mathrm{dim} X$.

Proof. It is clear that if $M \subset A_s$ is a compact subvariety then $\dim M \leq \dim X$ (The intersection of M with any fiber of the projection map $p : A_s \to X$ will be at most 0-dimensional). To exclude the case $\mathrm{dim} M = \mathrm{dim} X$, we will show that $p|_M^* s \in H^1(M, p^*V)$ is trivial and then show that if $\mathrm{dim} M = \mathrm{dim} X$ a variant of lemma 2.1 implies that $p|_M^* s$ is nontrivial.

To show that $p|_M^* s \in H^1(M, p^*V)$ is trivial, we construct a splitting of pullback to M of the exact sequence (3.1). The subvariety M has canonically associated with it a vector subbundle $W_M \subset V_s^*$, with $\mathrm{rk}\, W_M = \mathrm{rk}\, V_s^* - 1$, such that the restriction of the surjection $q : V_s^* \to V^*$ to W_M is still a surjection and therefore an isomorphism. The splitting of $0 \to \mathcal{O} \to p|_M^* V_s^* \to p|_M^* V \to 0$ is obtained by inverting $q : W_M \to V^*$.

If dim$M$ = dim$X$ the map $p|_M : M \to X$ would be a finite map and the following lemma 3.1 would imply that $p|_M^* s \in H^1(M, p^*V)$ is nontrivial.

Lemma 3.1. *Let $f : Y \to X$ a finite morphism between projective varieties X and Y with X nonsingular and V a vector bundle over X. If $s \in H^1(X, V)$ is nontrivial then $f^*s \in H^1(Y, f^*V)$ is also nontrivial.*

Proof. If $\dim X = 1$ then f is flat ([Ha77], III.9.7) and the result follows from lemma 2.1. If $\dim X > 1$, we use a dimension cutting argument. Let H be an hyperplane section of X, from the exact sequence $0 \to V \otimes \mathcal{O}(-nH) \to V \to V|_{nH} \to 0$ and $H^1(X, V \otimes \mathcal{O}(-nH)) = 0$ for $n \gg 0$, it follows that $H^1(X, V) \hookrightarrow H^1(H', V|_{H'})$, for a smooth hyperplane section $H' \in |nH|$. Repeat the argument for the smooth H' and then continue cutting dimensions till a smooth curve C is obtained. Since injectivity occurs in each step it follows that $H^1(X, V) \hookrightarrow H^1(C, V|_C)$.

Given a nontrivial cocycle $s \in H^1(X, V)$ by restriction we obtain the nontrivial cocycle $s|_C \in H^1(C, V|_C)$ for the curve C defined above. Since map $f|_{f^{-1}(C)} : f^{-1}(C) \to C$ is a flat map, lemma 2.1 implies that $f^*s|_{f^{-1}(C)} \in H^1(f^{-1}(C), f^*V|_{f^{-1}(C)})$ is nontrivial and therefore $f^*s \in H^1(Y, f^*V)$ must be nontrivial. ☐

☐

(continuation of proof of theorem 3.1) The holomorphic function $q \in \mathcal{O}(\tilde{X})$ satisfying $q(\rho^{-1}(D)) = 0$ and $q(\rho^{-1}(a)) = 1$ can be obtained by pulling back, using g, a holomorphic function $q' \in \mathcal{O}(A_s)$ that satisfies $q'(p^{-1}(D)) = 0$ and $q'(p^{-1}(a)) = 1$. The existence of such q' follows from $St(A_s)$ being Stein and $r : A_s \to St(A_s)$ being a proper map. ☐

It is important to complement theorem 3.1 with an example that shows that the hypothesis of the theorem do not imply that the universal cover $\tilde{X}$ is Stein (if the affine bundle A_s is Stein then in this case $\tilde{X}$ is Stein, see corollary 3.2).

Example: Let X be a nonsingular projective variety whose universal cover $\tilde{X}$ is Stein. Let $\sigma : Y \to X$ be the blow up of X at a point $p \in X$, $E = \sigma^{-1}(p) = \mathbb{P}^{n-1}$, $n = \dim X$. If we pullback the exact sequence (2.5) to Y and tensor it with $\mathcal{O}(E)$ we obtain:

$$0 \to \sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E) \to \bigoplus^{n+1} \sigma^* L^{-1} \otimes \mathcal{O}(E) \to \mathcal{O}(E) \to 0 \tag{3.2}$$

The vector bundle $\sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E)$ on Y is negative since $\sigma^* L^{-1} \otimes \mathcal{O}(E)$ is a negative line bundle on Y. The pair Y and $\sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E)$ satisfies the conditions of theorem 3.1 but $\tilde{Y}$ is clearly not Stein (it contains $\pi_1(X)$ copies of $\mathbb{P}^{n-1}$).

Example: We also give an example of a projective variety X whose universal cover is Stein having a negative vector bundle with a nontrivial cocycle $s \in H^1(X, V)$ that becomes trivial when restricted to a non-exceptional subvariety $Z \subset X$. Let X be Kodaira fibration $f : X \to C$ (non-isotrivial smooth fibration of curves over a smooth curve C). These surfaces were constructed by Kodaira [Ko67], have a negative tangent bundle T_X

[Sc86] and have a Stein universal cover [Sb77]. Choose a Kodaira fibration X with fibers of genus g but a base of sufficient large genus such that $H^1(X, T_X) > 4g-3$ and hence there is a nontrivial cocycle $s \in H^1(X, T_X \otimes \mathcal{O}(-F))$ since $H^1(F, T_X|_F) \leq 4g-3$. .

The method used in the proof of theorem 3.1 shows also that:

Corollary 3.1. *Let X be a projective variety with a negative vector bundle V and $\rho : \tilde{X} \to X$ its universal covering. If it exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^* s = 0$ then $\dim_h \tilde{X} = \dim_{Sh} X = \dim_{\mathbb{C}} X$ and the holomorphic functions on $\tilde{X}$ separate points on $\tilde{X} \setminus \rho^{-1}(Z)$, Z a subvariety of X.*

Proof. In the proof of theorem 3.1 it was constructed a holomorphic map $g = \rho' \circ s : \tilde{X} \to A_s$ giving a local biholomorphism between $\tilde{X}$ and $g(\tilde{X}) \subset A_s$. Moreover, $g(\tilde{X}) \subsetneq M$ where M is the maximal compact subset of the strongly pseudoconvex space A_s, therefore $\dim_h \tilde{X} = \dim_{\mathbb{C}} X$ and the result follows from Proposition 1.1.

Let a and b be two different points in $\tilde{X}$ and $b \notin \rho^{-1}(Z)$ with Z as in the theorem. The $\bar{\partial}$-method gives that for a positive line bundle L on X the line bundle $\rho^* L^p$ on $\tilde{X}$ has a section s with $s(a) = 0$ and $s(b) \neq 0$ for $p \gg 0$ [Na90]. Let $s' \in H^0(X, L^p)$ be such that $\rho(b) \notin D = (s')_0$. The meromorphic function $h = \frac{s}{\rho^* s'}$ is holomorphic outside $\rho^{-1}(D)$ and $h(b) \neq 0$. As shown in the proof of theorem 3.1 there is a $q \in \mathcal{O}(\tilde{X})$ vanishing on $\rho^{-1}(D)$ and not vanishing at $\rho^{-1}(b)$. Then for l sufficiently large $f = hq^l$ would be the desired holomorphic function. □

In section 2.1 it was shown that $\dim_h \tilde{X} \leq \dim_{Sh} X$. Corollary 3.1 shows that $\dim_h \tilde{X} = \dim_{Sh} X$ under the hypothesis of the theorem. This motivates the following question: does $\dim_h \tilde{X} = \dim_{Sh} X$ hold for any projective variety X?

Let us consider the special case where the affine bundle A_s over X is a Stein manifold. The condition that A_s is an Stein manifold can be easily fulfilled in examples. Over $\mathbb{P}^n$ we have that the affine bundle F associated with the extension $0 \to \Omega^1_{\mathbb{P}^n} \to \bigoplus^{n+1} \mathcal{O}(-1) \to \mathcal{O} \to 0$ is isomorphic to the affine variety:

$$\mathbb{P}^n \times \mathbb{P}^{n\vee} \setminus \{(x, h) \in \mathbb{P}^n \times \mathbb{P}^{n\vee} | x \in h\}$$

Let X is a projective variety embedded in $\mathbb{P}^n$ and F_X the affine bundle associated with pullback to X of the above extension. F_X is a Stein manifold since it is a closed subvariety of F. We give corollary of theorem 3.1 for these special cases.

Corollary 3.2. *Let X, V and s be as in Theorem 3.1. Assume furthermore that A_s is a Stein variety. Let $f : Y \to X$ be any infinite unramified covering s.t. $f^* s = 0$. Then Y is Stein.*

Proof. Since any non-ramified covering of a Stein space is Stein [5] the assumption that A_s is affine yields that $A_s \times_X Y$ is Stein. On the other hand in the proof of Theorem 3.1 we saw that $Y \subset A_s \times_X Y$ is a closed analytic subset and so Y is Stein. □

Corollary 3.2 suggests that the result of Theorem 3.1 may also be applicable to orbicoverings of X. Let us first describe precisely the notion of orbicovering in the case of a complex variety. Let X be a complex variety and $S \subset X$ be a proper analytic subset. Consider for any point $q \in S$ the local fundamental group $\pi_q = \pi_1(U(q) \setminus S)$ where $U(q)$ is a small ball in X centered at q. Let $L \subset \pi_1(X \setminus S)$ be a subgroup with the property that $L \cap \pi_q$ is of finite index in π_q for all $q \in S$. Then the nonramified covering of $X \setminus S$ corresponding to L can be naturally completed into a normal complex variety Y_L with a locally finite and locally compact surjective map $f_L : Y_L \to X$. The map $f_L : Y_L \to X$ is called an *orbicovering* of X with a ramification set S. The following holds:

Corollary 3.3. *Let X, V and s be as in Theorem 3.1. Assume furthermore that A_s is an affine variety. Let $f : Y \to X$ be any orbicovering s.t. $f^*s = 0$. Then Y is Stein.*

Proof. Since every orbicovering of a Stein space is also Stein (see Theorem 4.6 of [5]) the proof is exactly the same as the proof of Corollary 3.2. ☐

Remark: It is worth to point out that in many of the cases for which the Shafarevich conjecture is known the proof relies on the comparison of two vector bundles on X which become equal when they are pulled back to $\widetilde{X}$ [3]. For example the theorem of M.Gromov uses the fact that some positive line bundle $E \to X$ becomes trivial when pulled back to the universal cover. The theorem of L.Katzarkov [4] establishes the holomorphic convexity of $\widetilde{X}$ for projective surface X under the assumption of the existence of an almost faithful linear representation of $\pi_1(X)$. In this case all the bundles on X corresponding to the representations of the fundamental group of the same dimension are becoming equal on $\widetilde{X}$.

Remark: In the case of surfaces the space $\widetilde{X}$ can be obtained as a union of two rather simple Stein manifolds with Stein intersection. This implies that the structure of the space of the moduli space of vector bundles on $\widetilde{X}$ in this case is somewhat similar to the structure of the moduli space of vector bundles on a curve. Namely any bundle of rank greater than 2 has a complete flag of subbundles, thus reducing the K-group $K_0(\widetilde{X})$ to $\mathrm{Pic}(X) \times \mathbb{Z}$. In particular one expects that many different bundles on X coincide after lifting to $\widetilde{X}$.

3.2 Semi-negative vector bundles (to be augmented soon).

Theorem 3.1 can not be applied to prove the above conjecture for all projective varieties. Corollary 3.1 shows that the hypothesis of theorem 3.1 implies $\dim_{Sh} X = \dim_{\mathbb{C}} X$ which does not always hold (among projective varieties of dimension n one can find varieties X with $\dim_{Sh} X = 0, ..., n$). To tackle the cases with $\dim_S X < \dim_{\mathbb{C}} X$ one has to consider semi-negative bundles.

Theorem 3.2. *Let X be a projective variety with a k-negative vector bundle V and $\rho : \tilde{X} \to X$ its universal covering. If it exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^* s = 0$ then the holomorphic dimension of $\tilde{X}$ satisfies $\dim_h \tilde{X} \geq \dim_{\mathbb{C}} X - k$.*

Proof. Let $s \in H^1(X, V)$ be as in the hypothesis and V_s the extension of V associated with s. Let A_s be the affine bundle $\mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ and $A_{\rho^* s}$ be the pullback of A_α to $\tilde{X}$ then $A_{\rho^* \alpha} \simeq \rho^*(V)$ since $\rho^* \alpha = 0$. $A_{\rho^* s}$ comes with an action of $\Gamma = \pi_1(X)$ such that $A_{\rho^* s}/\Gamma = A_s$. This action seen as acting on $\rho^*(V)$ can not preserve the zero section $Z(\rho^* V)$. This implies that $\rho^* V$ has nonzero sections $\gamma(Z(\rho^* V))$, $\gamma \in \Gamma$. Using these nonzero sections and covering map of $t(V)$ $\rho' : t(\rho * V) \to t(V)$ we obtain holomorphic maps $g_\gamma : \tilde{X} \to t(V)$ such that $g_\gamma(\tilde{X}) \not\subset Z(V)$.

Proposition 2.2 states that there is a proper holomorphic map $r : t(V) \to \mathbb{C}^l$ with at most k-dimensional fibers apart from $Z(t(V))$. The holomorphic dimension $\dim_h \tilde{X} \geq \dim_{\mathbb{C}} X - k$ follows since $g_\gamma : \tilde{X} \to g_\gamma(\tilde{X})$ is locally a biholomorphism. □

3.3 Special cycles.

Definition 3.3. A subvariety $Z \subset X$ will be called π_1-small if for some (and therefore every) resolution of singularities $res(Z)$ of Z the image $im[\pi_1(res(Z)) \to \pi_1(X)]$ is a finite group.

The second author motivated by the his work with L.Katzarkov and M. Ramachandran [6] has raised the following question.

Question. *Let X be a projective variety with infinite fundamental group. Are all the maximal π_1-small subvarieties of X contained in a finite collection of algebraic equivalence classes?*

An affirmative answer to this question is necessary for the Shafarevich conjecture to hold. Also, note that this question includes a conjecture of Lang, stating that a surface of general type has at most finite number of rational curves, here with the extra condition that the fundamental group of the surface is infinite (note that rational curves are necessarily π_1-small). To tackle this question we single out the following lemma that is implicitly present in the proof of the theorem.

Lemma 3.3. *Let V be a negative bundle on a projective variety X and let $s \in H^1(X, V)$. Then there exists a finite set of projective schemes $\{M_i\}_{i \in I}$, $\dim_{\mathbb{C}} M_i < \dim_{\mathbb{C}} X$ and affine morphisms $f_i : M_i \to X$ with the following property. Let Y be an irreducible projective variety and let $f : Y \to X$ be a map with $f^* s = 0$ such that $f(Y)$ is not a point. Then there exists a $i \in I$, such that f factors as $f = f_i \circ g$ where $g : Y \to M_i$.*

Proof. Indeed using the notation of the proof of Theorem 3.1 we have that $f^* V_s^*$ splits into a direct sum $f^* V^* \oplus \mathcal{O}_Y$. Hence $A_{f^* s} \equiv Y \times_X A_s \simeq f^* V$ contains Y as a closed subvariety, denote by $j : Y \to A_{f^* s}$ the embedding of Y as the zero section.

Consider the projection $p_2 : Y \times_X A_s \to A_s$ and denote by $g : Y \to A_s$ the composition of j with p_2. As described in the proof of Theorem 3.1 all compact subvarieties of A_s are contained in a finite collection of subvarieties $M = \cup_{i=1}^{k} M_i$. Hence $g(Y) =\subset M_i$ for some i and therefore $f = f_i \circ g$ which proves the lemma. $\square$

Remark 1. As the proof shows the test variety Y in the above lemma can be any connected variety without non constant holomorphic functions.

Corrolary 3.4. *If X and V satisfy the conditions of Theorem 3.1 then all π_1-small subvarities of X are contained in a finite union of subvarieties of X.*

Proof:. If Z is a π_1-small subvariety then its preimage is compact or it is an infinite chain of compact subvarieties. Denote by $Z_i' \subset \tilde{X}$ any compact component of the preimage of Z. Z_i' is a projective variety with a map $\rho|_{Z_i'} : Z_i' \to X$ such that $\rho|_{Z_i'}^{*} s = 0$, and the result follows from Lemma 3.1. $\square$

4. Geometrical vanishing theorem for negative bundles

The arguments used in the proof of Theorem 3.1 can be used to give an alternative proof of the vanishing theorem for negative vector bundle V over a projective variety X whose $\operatorname{rk} V < \dim X$.

Theorem 4.1. *If V is a negative vector bundle on X with $\operatorname{rk} V < \dim X$, then $H^1(X, V) = 0$.*

Proof:. Suppose it exists a nontrivial $s \in H^1(X, V)$ and let:

$$0 \to V \to V_s \to \mathcal{O} \to 0$$

be the associated extension. As in the Theorem 3.1, consider the dual exact sequence and $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ be an affine bundle, which by the negativity of V is strictly pseudoconvex. Let $r : A_s \to St(A_s)$ be the Remmert reduction, where r is proper contracting $M = \cup_{i=1}^{k} M_i$ and $St(A_s)$ is a Stein space with isolated singularities.

The aim is to obtain a contradiction from topological conditions. The Stein space $St(A_s)$ has $\dim_{\mathbb{C}} St(A_s) = \dim_{\mathbb{C}} X + r$ and hence it has the homotopy type of a simplicial complex of real dimension at most equal to $\dim_{\mathbb{C}} X + r$. On the other hand, $St(A_s) = A_s/(\coprod_i M_i)$ as a topological space and so for the reduced singular homology of A_s $\widetilde{H}_i(St(A_s), \mathbb{C}) = H_i((A_s, \coprod_i M_i), \mathbb{C})$. Now the long exact homology sequence of the pair $(A_s, \coprod_i M_i)$ together with the fact that $\coprod_i M_i$ is compact of complex dimension strictly less than $\dim_{\mathbb{C}} X = n$ gives that $H_{2n}(A_s, \mathbb{C}) \cong \widetilde{H}_{2n}(St(A_s), \mathbb{C}) = H_{2n}(St(A_s), \mathbb{C})$.

In conclusion, $St(A_s)$ as a Stein manifold of $\dim_{\mathbb{C}} St(A_s) = n + r < 2n$ must have $H_{2n}(St(A_s), \mathbb{C}) = 0$ but at the same time using the previous argument $H_{2n}(St(A_s), \mathbb{C}) \cong H_{2n}(A_s, \mathbb{C})$. The contradiction follows since A_s as an affine bundle over X is homotopicaly equivalent to X and therefore $H_{2n}(A_s, \mathbb{C}) \cong H_{2n}(X, \mathbb{C}) \neq 0$.

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