SYMMETRIC TENSORS AND THE GEOMETRY OF SUBVARIETIES OF $\mathbb{P}^N$.

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0. Introduction

This paper following a geometric approach proves new, and reproves old, vanishing and nonvanishing results on the space of twisted symmetric differentials, $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(k))$ with $k \leq m$, on subvarieties $X \subset \mathbb{P}^N$. The case of $k = m$ is special and the nonvanishing results are related to the space of quadrics containing X and lead to interesting geometrical objects associated to X, as for example the variety of all tangent trisecant lines of X. The same techniques give results on the symmetric differentials of subvarieties of abelian varieties. The paper ends with new results and examples about the jump along smooth families of projective varieties X_t of the symmetric plurigenera, $Q_m(X) = \dim H^0(X, S^m\Omega^1_X)$, or of the α-twisted symmetric plurigenera, $Q_{\alpha,m}(X) = \dim H^0(X, S^m(\Omega^1_X \otimes \alpha K_X))$.

The paper was in part motivated to answer the question of M.Paun about whether $\dim H^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ is locally invariant in smooth families if $K_{X_t} > 0$. This invariance would be the natural extension of the result of Y-T. Siu on the invariance of plurigenera to other tensors. The negative answer appears in the last section. In a previous result we proved that while smooth hypersurfaces in $\mathbb{P}^3$ do not have symmetric differentials, resolutions of nodal hypersurfaces have them if the number of nodes is sufficiently large. This is, in particular, interesting because smooth and resolutions of nodal hypersurfaces in $\mathbb{P}^3$of the same degree are deformation equivalent. So we have a case of the jumping of the symmetric plurigenera $Q_m(X)$ and a special one for that matter, as we shall see below. There is a previous example of this phenomenon in [Bo2-78], see section 2. Recall that this contrasts with the invariance of the plurigenera $P_m = \dim H^0(X, (\wedge^n\Omega^1_X)^m)$, $n = \dim X$, [Si98]. The jumping in our example might bring back symmetric differentials to new approaches to the Kobayashi's conjecture which states that a general hypersurace in $\mathbb{P}^3$ of degree $d \geq 5$ is hyperbolic (the known approaches use jet differentials, i.e. higher order symmetric differentials, which exist on hypersurfaces but

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are quite difficult to control). The authors motivated by this unexpected appearance of the symmetric differentials realized that there are many unanswered or forgotten questions about them.

P. Bruckman showed in [Br71] that there are no symmetric differentials on smooth hypersurfaces in $\mathbb{P}^N$ via an explicit constructive approach. Later, F. Sakai with a cohomological approach using a vanishing theorem of Kobayashi and Ochai showed that a complete intersection $Y \subset \mathbb{P}^N$ with dimension $n > N/2$ has no symmetric differentials [Sa78]. In the early nineties M. Schneider [Sc92] using a similar approach, but with more general vanishing theorems of le Potier, showed that any submanifold $X \subset \mathbb{P}^N$ of dimension $n > N/2$ has no symmetric differentials of order m even if twisted by $\mathcal{O}_X(k)$, $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(k)) = 0$, where $k < m$.

In this paper we use a distinct approach to obtain vanishing and nonvanishing results on the space of twisted symmetric differentials that are the natural extension of the results mentioned above. Parts of this approach can be traced back to an announcement in the ICM of 1978 by the first author, [Bo78]. Our method has a geometric flavor involving of the tangent map for X. The tangent map is given by $f : \mathbb{P}(\widetilde{\Omega^1_X(1)}) \to \mathbb{P}^N$, where $f(\mathbb{P}(\widetilde{\Omega^1_X(1)})_x) = T_xX$ and T_xX is the embedded projective tangent space to X at x in $\mathbb{P}^N$. The first application of our approach is to show that if the tangent map for X has a positive dimensional general fiber, then X has no symmetric differentials of order m even if twisted by $\mathcal{O}_X(m) \otimes L$, where L is any negative line bundle on X, i.e. $H^0(X, S^m[\Omega^1_X(1)] \otimes L) = 0$ (this includes Schneider's result).

The next step in the paper is to analyse the case of symmetric powers of the sheaf of twisted differentials , i.e. $S^m[\Omega^1_X(1)]$. This would be the case with $L = \mathcal{O}_X$ in the result just mentioned above or equivalently the case $k = m$ not reached by Schneider's methods. The loss of the negatitivity of L makes the existence of twisted symmetric differentials more delicate. We use the pivotal lemma 1.1 about the sections of symmetric powers of quotients of trivial vector bundles. One of the requirements to use the lemma is that the tangent map associated to $X \subset \mathbb{P}^N$ must be surjective and connected. A key result, theorem 1.3, is that connectedness of the fibers of the tangent map is guaranteed if $dim X > 2/3(N-1)$.

An, perhaps surprisingly, important feature of the existence of twisted symmetric differentials is that it depends on the t-trisecant variety of X. The t-trisecant variety of X is the subvariety of the trisecant variety of X consisting of the union of the tangent lines to X which are also trisecant lines. In fact, the answer might also depend on of higher level t-trisecant varieties of X, see section 1.2.

We show in theorem C that if $X \subset \mathbb{P}^N$ has dimension $n > 2/3(N-1)$, then $H^0(X, S^m[\Omega^1_X(1)]) \neq 0$ if and only if all higher order t-trisecant varieties X are not $\mathbb{P}^N$. This holds in particular if X is contained in a quadric. When X is of codimension 1 or 2 and of dimension $n \geq 2$ $n \geq 3$, respectively, the result is $H^0(X, S^m[\Omega^1_X(1)]) \neq 0$ if and only X is contained in a quadric (the codimension 2 case is the more challenging case). Here an important characteristic is that one only needs to consider the t-trisecant variety. Moreover, it is shown that in these cases the t-trisecant variety is the trisecant variety. We also give a general criterion of when the t-trisecant and the trisecant variety of X coincide. As an application of the circle ideas behind this criterion we give an alternative proof of the Zak's theorem on the equality $Tan(X) = Sec(X)$ if $Sec(X)$ does not have the expected dimension.

In the last section we answer a question of M. Paun asking if $\dim H^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ is locally invariant in smooth families if $K_{X_t} > 0$. We give a negative answer based on the results on nonexistence twisted symmetric on hypersurfacesthe plus the example, mentioned above, of the family of smooth hypersurfaces in $\mathbb{P}^3$ specializing to the resolution of a nodal surfaces with sufficiently many nodes. We then ask whether there is a ratio k/m for which the invariance of $\dim H^0(X_t, S^m\Omega^1_{X_t} \otimes (\wedge^n\Omega^1_X))^k)$ holds and address the question of what is the lowest of such ratios.

1. Symmetric differentials on subvarieties of $\mathbb{P}^N$ and of Abelian varieties

1.1 Preliminaries.

Let E be a vector bundle on X and $\mathbb{P}(E)$ be the projective bundle of hyperplanes of E. Recall the connection between S^mE and $\mathcal{O}_{\mathbb{P}(E)}(m)$ which plays a fundamental role in the study of symmetric powers of a vector bundle. If $\pi : \mathbb{P}(E) \to X$ is usual projection map then the following holds $\pi_*\mathcal{O}_{\mathbb{P}(E)}(m) \cong S^mE$ and

$$H^0(X, S^mE) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \tag{1.1.1}$$

The following case persistently appears in our arguments. Let E be a vector bundle on X which is a quotient of $\bigoplus^{N+1} L$ where L is a line bundle on X:

$$q : \bigoplus^{N+1} L \to E \to 0 \tag{1.1.2}$$

Let $\mathbb{P}(E)$ and $\mathbb{P}(\bigoplus^{N+1} L)$ be the projective bundles of hyperplanes of E and $\bigoplus^{N+1} L$ respectively. The surjection q induces an inclusion and the isomorphism:

$$i_q : \mathbb{P}(E) \hookrightarrow \mathbb{P}(\bigoplus^{N+1} L)$$
$$i_q^*\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$$

Recall that there is a natural isomorphism $\phi : \mathbb{P}(\bigoplus^{N+1} L) \to \mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ for which $\phi^*\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O})}(1) \cong \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \otimes \pi^*L^{-1}$. The projective bundle $\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ is the product $X\times\mathbb{P}^N$, if p_2 denotes the projection onto the second factor, then $\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1}\mathcal{O})}(1)\cong p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$. Concluding, the surjection q in (2) naturally induces a map $f_q = p_2\circ\phi\circ i_q$ and the isomorphism:

$$f_q : \mathbb{P}(E)\to\mathbb{P}^N$$

$$f_q^*\mathcal{O}_{\mathbb{P}^N}(1)\cong\mathcal{O}_{\mathbb{P}(E)}(1)\otimes\pi^*L^{-1} \tag{1.1.3}$$

Hence

$$H^0(X, S^mE)\cong H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m)\otimes\pi^*L^{\otimes m}) \tag{1.1.4}$$

It follows from (1.1.4) that the properties of the map $f_q : \mathbb{P}(E)\to\mathbb{P}^N$ have an impact on the existence of sections of the symmetric powers of E. The next result gives an example of this phenomenon and will play a role in our study of existence of symmetric differentials.

Lemma 1.1. *Let E be a vector bundle on a smooth projective variety X. If E is the quotient of a trivial vector bundle:*

$$q : \bigoplus^{N+1}\mathcal{O}_X\to E\to 0$$

and the induced map $f_q : \mathbb{P}(E)\to\mathbb{P}^N$ is surjective with connected fibers, then q induces the isomorphism:

$$H^0(X, S^mE) = H^0(X, S^m(\bigoplus^{N+1}\mathcal{O}_X))$$

($H^0(X, S^mE) = S^m[\mathbb{C}s_0\oplus ...\oplus\mathbb{C}s_N]$ where $s_i = q(e_i)$, $\bigoplus^{N+1}\mathcal{O}_X = \bigoplus_{i=0}^N\mathcal{O}_Xe_i$).

Proof. The isomorphism $f_q^*\mathcal{O}_{\mathbb{P}^N}(m)\cong\mathcal{O}_{\mathbb{P}(E)}(m)$, (1.1.3), and $H^0(X, S^mE)\cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m))$ give that:

$$H^0(X, S^mE)\cong H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m))$$

The next step is to relate $H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m))$ with $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$. If f_q is surjective then $f_q^* : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))\to H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m))$ is injective. If the map f_q also has connected fibers, then all sections in $H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m))$ descend to be sections in $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$, and the following holds:

$$H^0(\mathbb{P}(E), f_q^*\mathcal{O}_{\mathbb{P}^N}(m))\cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

The result then follows from the brake down of the map f_q, $f_q = p_2\circ i_q$, plus $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))\cong H^0(\mathbb{P}(\bigoplus^{N+1}\mathcal{O}_X), p_2^*\mathcal{O}_{\mathbb{P}^N}(m))\cong H^0(\mathbb{P}(\bigoplus^{N+1}\mathcal{O}_X), \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1}\mathcal{O}_X)}(m))$ and $H^0(\mathbb{P}(\bigoplus^{N+1}\mathcal{O}_X), \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1}\mathcal{O}_X)}(m))\cong H^0(X, S^m(\bigoplus^{N+1}\mathcal{O}_X))$. $\square$

1.2 Symmetric differentials on subvarieties of $\mathbb{P}^N$.

The following is short collection of facts about the sheaf of differentials that will help the reader understand our approach. The Euler sequence of $\mathbb{P}^N$ is:

$$0 \to \Omega^1_{\mathbb{P}^N} \to \bigoplus^{N+1} \mathcal{O}(-1) \to \mathcal{O} \to 0 \tag{1.2.1}$$

The Euler sequence expresses the relation, induced by the natural projection $p : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$, between the differentials of $\mathbb{C}^{N+1}$ and $\mathbb{P}^N$. A necessary condition for a differential ω of $\mathbb{C}^{N+1}$ to come from a differential of $\mathbb{P}^N$ is that the coefficients $h_0(z)$,...,$h_N(z)$ of $\omega = h_0(z)dz_0 + ... + h_N(z)dz_N$ must be homogeneous of degree -1. But the last condition is not sufficient, the differentials ω on $\mathbb{C}^{N+1}$ must be such that at any point $z \in \mathbb{C}^{N+1}$ their contraction with the vector $z_0\partial/\partial z_0 + ... + z_N\partial/\partial z_N$, i.e with the direction of the line from z to the origin, must be zero. To see this algebraically, the sheaf $\bigoplus^{N+1} \mathcal{O}(-1)$ in (1.2.1) is $\bigoplus^{N+1} \mathcal{O}(-1) = \mathcal{O}(-1)dz_0 + ... + \mathcal{O}(-1)dz_N$. The map $q : \bigoplus^{N+1} \mathcal{O}(-1) \to \mathcal{O} \to 0$ is defined sending dz_i to z_i. So locally, let us say on $U_i = \{z_i \neq 0\}$, $\Omega^1_{U_i}$ the kernel of the map q is spanned by the sections induced by the differentials $\frac{1}{z_i}dz_j - \frac{z_j}{z_i^2}dz_i$ on $p^{-1}(U_i)$.

The sheaf of differentials Ω^1_X is determined by (1.2.1) restricted to X:

$$0 \to \Omega^1_{\mathbb{P}^N}|_X \to \bigoplus^{N+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \tag{1.2.2}$$

and the conormal bundle exact sequence:

$$0 \to N^* \to \Omega^1_{\mathbb{P}^N}|_X \to \Omega^1_X \to 0 \tag{1.2.3}$$

The extension defined by (1.2.2) (which corresponds to a cocycle $\alpha \in H^1(X, \Omega^1_{\mathbb{P}^N}|_X)$) induces via the surjection in (1.2.3) the extension:

$$0 \to \Omega^1_X \to \widetilde{\Omega^1_X} \to \mathcal{O}_X \to 0 \tag{1.2.4}$$

The geometric description of the sheaf $\widetilde{\Omega^1_X}$ is that it is the sheaf on X associated to the sheaf of 1-forms on the affine cone $\hat{X} \subset \mathbb{C}^{N+1}$. The above exact sequences after twisted by $\mathcal{O}_X(1)$ fit in the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1_{\mathbb{P}^N}|_X(1) \arrow[r] \arrow[d] & \bigoplus^{N+1} \mathcal{O}_X \arrow[r] \arrow[d, "q"'] & \mathcal{O}_X(1) \arrow[r] \arrow[d, "\simeq"'] & 0 \\
0 \arrow[r] & \Omega^1_X(1) \arrow[r] & \widetilde{\Omega^1_X}(1) \arrow[r] & \mathcal{O}_X(1) \arrow[r] & 0
\end{tikzcd} \tag{1.2.5}$$

The middle vertical surjection of diagram (1.2.5) can be represented more explicitly by:

$$q : \bigoplus_{i=0}^{N} \mathcal{O}_X dz_i \to \widetilde{\Omega^1_X}(1) \tag{1.2.6}$$

The induced map $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$ is such that for each $x \in X$:

$$f(\mathbb{P}(\widetilde{\Omega^1_X}(1))_x) = T_x X \tag{1.2.7}$$

where $T_x X$ is the embedded projective tangent space to X at x inside $\mathbb{P}^N$. For the obvious reasons f will be called the tangent map for X. The tangent map f induces a map from X to $G(n, N)$ which is exactly the Gauss map for X, $\gamma_X : X \to G(n, N)$.

Theorem A. *Let X be a smooth projective subvariety of $\mathbb{P}^N$. If the general fiber of the tangent map for X, $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$, is positive dimensional, then $\forall m \geq 0$:*

$$H^0(X, S^m[\Omega^1_X(1)] \otimes L) = 0$$

if L is a negative line bundle on X.

Proof. It is sufficient to show that $H^0(X, S^m[\widetilde{\Omega^1_X}(1)] \otimes L) = 0$, since there is the inclusion $S^m[\Omega^1_X(1)] \hookrightarrow S^m[\widetilde{\Omega^1_X}(1)]$, induced from (1.2.5).

The projective bundle $\mathbb{P}(\widetilde{\Omega^1_X}(1))$ comes with two maps. The tangent map for X, $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$, and the projection onto X, $\pi : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to X$. One also has the natural isomorphisms $\mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m) = f^*\mathcal{O}_{\mathbb{P}^N}(m)$ and $\pi_*(\mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m) \otimes \pi^* L) \cong S^m[\widetilde{\Omega^1_X}(1)] \otimes L$. These isomorphisms give:

$$H^0(X, S^m[\widetilde{\Omega^1_X}(1)] \otimes L) \cong H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L)$$

The vanishing of the last group follows from the negativity of the line bundle $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L$ along each fiber of the map f. More precisely, $f^*\mathcal{O}_{\mathbb{P}^N}(m)$ is trivial on the fibers and $\pi^* L$ is negative on the fibers since L is negative on X the map π is injective on each fiber of f.

We need the fibers of the map f to be positive dimensional. Since is only in this case that the negativity of the line bundle $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L$, $l < 0$, makes sense. This negativity implies that all sections of $H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^* L)$ vanish along all fibers of f and hence vanish on all $\mathbb{P}(\widetilde{\Omega^1_X}(1))$, which completes the proof. $\square$

As an important case of theorem A one has another proof to the result first proved by Schneider [Sc92].

Corollary 1.2. *Let X be a smooth projective subvariety of $\mathbb{P}^N$ whose dimension $n > N/2$. Then:*

$$H^0(X, S^m\Omega^1_X \otimes \mathcal{O}(k)) = 0$$

if $k < m$.

Proof. The dimensional hypothesis $n > N/2$ guarantee that all fibers of the tangent map f for X are positive dimensional. The condition $k < m$ gives that $S^m\Omega^1_X \otimes \mathcal{O}(k) = S^m[\Omega^1_X(1)] \otimes \mathcal{O}_X(l)$, with $l < 0$. The theorem then follows from theorem A for the negative line bundle $L = \mathcal{O}_X(l)$, $l < 0$. □

What happens in the key case $k = m$? The results just mentioned use the negativity $f^*\mathcal{O}_{\mathbb{P}^N}(m) \otimes \pi^*L$, along the fibers of the map f, which no longer holds if $k = m$. Indeed, one has $H^0(X, S^m\widetilde{\Omega^1_X} \otimes \mathcal{O}_X(m)) = H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), f^*\mathcal{O}_{\mathbb{P}^N}(m))$ which is no longer trivial. The analysis of the nonexistence of twisted symmetric differentials $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ on X is more delicate. One has to describe the sections $H^0(X, S^m\widetilde{\Omega^1_X} \otimes \mathcal{O}_X(m))$ and characterize which ones are in $H^0(X, S^m\Omega^1_X \otimes \mathcal{O}_X(m))$. The answers will depend on geometric properties involving the variety of tangent lines to the subvariety X.

To describe the twisted symmetric extended differentials in $H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ one needs to use the properties of the tangent map for $X \subset \mathbb{P}^N$. The lemma 1.1 gives a good description of $H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ if the tangent map f is a connected surjection. The next lemma shows that this is the case when $\dim X > 2/3(N-1)$.

This paragraph about the tautological $\mathbb{P}^1$-bundle over the grassmanian is used in the lemma below. Let $q : I \to G(1, \mathbb{P}^n)$ be the tautological $\mathbb{P}^1$-bundle over the grassmanian and $p : I \to \mathbb{P}^n$ the natural map. For any point $x \in \mathbb{P}^n$ there is a $\mathbb{P}^{n-1} \subset G(1, \mathbb{P}^n)$ consisting of all lines passing through x. The restriction (or the pullback) of the tautological $\mathbb{P}^1$-bundle to $\mathbb{P}^{n-1}$ is $q : \mathbb{P}(\mathcal{O}(1) \oplus \mathcal{O}) \to \mathbb{P}^{n-1}$.

Theorem 1.3. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then the natural map $f : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to \mathbb{P}^N$ associated with the Gauss map is a surjective and connected morphism.*

Proof. The description of the map f in (1.2.7) implies that:

$$f(\mathbb{P}(\widetilde{\Omega^1_X}(1))) = Tan(X)$$

where $Tan(X)$ is the tangent variety of X, in other words $Tan(X) = \bigcup_{x \in X} T_xX \subset \mathbb{P}^N$. Denote by $Sec(X) \subset \mathbb{P}^N$ the secant variety of X. Zak's results [Za81] about tangencies state that one of the following must hold: i) $\dim Tan(X) = 2n$ and $\dim Sec(X) = 2n+1$; ii) $Tan(X) = Sec(X)$. It follows immediately that if $\dim X \geq N/2$ then $Tan(X) = Sec(X)$.

It is also a result of Zak, coming from applying the results on tangencies plus the Terracini's lemma on the tangent spaces of secant varieties, that $Sec(X) \neq \mathbb{P}^N$ can hold only if $n \leq 2/3(N-2)$. Hence surjectivity of f is guaranteed if $n > 2/3(N-2)$, which is the case.

It remains to show the connectedness of the fibers. Denote the fibers of f by $Y_x = f^{-1}(x)$ for $x \in \mathbb{P}^N$ and $\pi : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \to X$ be the projection map. The injectivity of f restricted to the fibers of π implies that Y_x is connected if and only if $\pi(Y_x)$ is connected. The subvariety $R_x = \pi(Y_x)$ is the locus of X consisting of all the points in X having a tangent line passing through x. The Stein factorization implies that f is connected if its general fiber is connected, i.e. if for the general $x \in \mathbb{P}^N$ the locus R_x is connected.

In the following arguments we always assume that $x \in \mathbb{P}^N$ is general. The first observation to make is that $R_x \subset Z_x$, where Z_x is the double locus of the projection $p_x : X \to \mathbb{P}^{N-1}$ (i.e. the locus of points in X belonging to lines passing through x and meeting X at least twice). By dimensional arguments one has that R_x is a Weyl divisor of Z_x. A key element in our argument is the result of [RaLo03] stating that the double locus Z_x is irreducible if $n > 2/3(N-1)$.

Let $S \subset \mathbb{P}^{N-1}$ be the image of Z_x by the projection p_x. The irreducible variety S can be seen as a subvariety of the $\mathbb{P}^{N-1} \subset G(1, \mathbb{P}^N)$ of lines passing through x. We can pullback the tautological $\mathbb{P}^1$-bundle on $G(1, \mathbb{P}^N)$ to S and obtain $q_S : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to S$. The natural map $p : I \to \mathbb{P}^N$, see the paragraph before the lemma, induces a map $p : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to \mathbb{P}^N$, whose image is the cone with vertex x and base S. The map p is a biregular morphism of the complement of $p^{-1}(x)$ onto the cone without the vertex.

The $\mathbb{P}^1$-bundle $q_S : \mathbb{P}(\mathcal{O}(1)|_S \oplus \mathcal{O}) \to S$ comes with two natural sections (one for each surjection onto the lines bundles $\mathcal{O}$ and $\mathcal{O}(1)$). Geometrically these two sections come from the pre-image of x and the pre-image of S via the map p. The subvariety $M = p^{-1}(Z_x)$ is biregular to Z_x and is a divisor in the total space of the line bundle $\mathcal{O}_S(1)$, $\mathbb{P}(\mathcal{O}_S(1) \oplus \mathcal{O}) \setminus \mathbb{P}(\mathcal{O})$. The points in $p^{-1}(R_x)$ are the points $y \in M$ for which the fibers of q_S meet M at y with multiplicity ≥ 2. The generality of x implies by the classical trisecant lemma that the general fiber of q_S meets M only twice counting with multiplicity. This makes the projection $q_S|_M : M \to S$ a generically 2 to 1 map.

Consider the pullback $L = q_S|_M^* \mathcal{O}_S(1)$ which is an ample line bundle on M. The line bundle L comes naturally with a nontrivial section denote the corresponding divisor of the total space of L, $Tot(L)$, by D_1. Denote the natural map between the total spaces of L and $\mathcal{O}_S(1)$ by $g : Tot(L) \to Tot(\mathcal{O}_S(1))$. The divisorial component of $g^{-1}(M)$ is decomposed in two irreducible components D_1 and D_2. Let $h : Tot(L) \to M$ be the natural projection, then $h(D_1 \cap D_2) \subset R_x$. If D_2 is also a section of L, then $h(D_1 \cap D_2)$ is connected since it is the zero locus $(s)_0$ of a section s of the ample line bundle L. The result would follow since any other possible component of R_x has to meet $(s)_0$. If D_2 is not a section the result stills follows from the same argument after base change (pulling back L to D_2 using h).

$\square$

In conjunction with lemma 1.1 one obtains the following description of the space of twisted extended symmetric differentials on X:

Corollary 1.4. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then:*

$$H^0(X, S^m[\widetilde{\Omega^1_X}(1)]) = S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N].$$

The characterization of the subset of $H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ consisting of the twisted symmetric differentials on X, within the dimensional range $\dim X > 2/3(N-1)$, is given by the following proposition:

Proposition 1.5. *Let X be a n-dimensional submanifold of $\mathbb{P}^N$ with $n > 2/3(N-1)$ then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = \{\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N] |\ Z(\Omega) \cap T_xX \textit{ is a cone with vertex at } x,\ \forall x \in X\}$$

Proof. The inclusion $H^0(X, S^m[\Omega^1_X(1)]) \subset H^0(X, S^m[\widetilde{\Omega^1_X}(1)])$ and corollary 1.4 imply that that all the sections of $H^0(X, S^m[\Omega^1_X(1)])$ are induced from the symmetric m-differentials $S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ on $\mathbb{C}^{N+1}$.

Let $\hat{X} \subset \mathbb{C}^{N+1}$ be the affine cone over $X \subset \mathbb{P}^N$, $T\hat{X}$ the sheaf on X associated with the tangent bundle of $\hat{X}$ and $T_xX \subset \mathbb{P}^N$ the embedded tangent space to X at x. Consider the rational map $p : \mathbb{P}(\widetilde{\Omega^1_X}(1)) \dashrightarrow \mathbb{P}(\Omega^1_X(1))$, which is fiberwise geometrically described by the projections from the point $x \in T_xX$ $p_x : T_xX = \mathbb{P}_l(T\hat{X})_x \dashrightarrow \mathbb{P}_l(TX)_x$, ($\mathbb{P}_l(E)$ is the projective bundle of lines in the vector bundle E, $\mathbb{P}_l(E) = \mathbb{P}(E^*)$). The map p gives an explicit inclusion $H^0(X, S^m[\Omega^1_X(1)]) = p^*H^0(\mathbb{P}(\Omega^1_X(1)), \mathcal{O}_{\mathbb{P}(\Omega^1_X(1))}(m)) \subset H^0(\mathbb{P}(\widetilde{\Omega^1_X}(1)), \mathcal{O}_{\mathbb{P}(\widetilde{\Omega^1_X}(1))}(m)) = S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$.

Recall that if $0 \to V \to \widetilde{V} \to \mathbb{C} \to 0$ is a sequence of vector spaces, then one gets a projection from $[V] \in \mathbb{P}(\widetilde{V})$, $p : \mathbb{P}(\widetilde{V}) \dashrightarrow \mathbb{P}(V)$. The sections in $H^0(\mathbb{P}(\widetilde{V}), \mathcal{O}_{\mathbb{P}(\widetilde{V})}(m))$ which are in $p^*H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(m))$ are the ones corresponding to homogeneous polynomials whose zero locus is a cone with vertex at $[V]$.

An element $\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ corresponds in a natural way to an homogeneous polynomial in $\mathbb{P}^N$ which we still denote by Ω. From the last two paragraphs it follows that Ω induces an element in $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ if and only if $\forall x \in X$ the zero locus $Z(\Omega) \cap T_xX$ is a cone with vertex x. $\square$

We proceed to extract from proposition 1.5 the geometric conditions required for the existence of twisted symmetric differentials on smooth subvarieties $X \subset \mathbb{P}^N$. First, we need to introduce some objects and notation.

Let X be an irreducible subvariety and Y be any subvariety of $\mathbb{P}^N$. Consider the incidence relation:

$$\mathcal{C}_X Y := \overline{\{(x,z) \in X_{sm} \times \mathbb{P}^N \mid z \in \overline{xy},\ y \neq x \text{ and } y \in Y \cap T_x X\}} \subset X \times \mathbb{P}^N$$

where X_{sm} denotes the smooth locus of X. The variety $\mathcal{C}_X Y$ comes with two projections. Denote by $C_X Y := p_2(\mathcal{C}_X Y)$. Equivalently, $\forall x \in X_{sm}$ denote by $C_x Y \subset T_x X$ the cone with vertex at x consisting of the closure of the union of all chords joining x to $y \neq x$ with $y \in Y \cap T_x X$, where $T_x X$ is the projective embedded tangent space to X at x. Then $C_X Y = \overline{\bigcup_{x \in X_{sm}} C_x Y} \subset \mathbb{P}^N$.

Definition 1.6. *Let X be an irreducible subvariety of $\mathbb{P}^N$. The variety $C_X X \subset \mathbb{P}^N$ will be called the t-trisecant variety of X.*

For hypersurfaces H, including singular but always reduced, one has the following useful result.

Proposition 1.7. *Let $H \subset \mathbb{P}^N$, $N > 2$, be an irreducible nondegenerate hypersurface. Then $C_H H = H$ if H is a quadric and $C_H H = \mathbb{P}^N$ otherwise.*

Proof. Let H be a quadric and $x \in H$ a smooth point. The lines $l \subset C_x H$ passing through x must touch H at least 3 times (counting with multiplicity) hence $l \subset H$. This implies that $C_x H \subset H$, for all $x \in H$, therefore $C_H H = H$.

Let H be of degree greater than 2. The result follows to the trivial case of curves in $\mathbb{P}^2$. For a general 2-plane L in $\mathbb{P}^N$ the intersection $H \cap L = D$ is an irreducible and reduced curve of the same degree as H. The irreducible and reduced curve $D = H \cap L$ of degree ≥ 3 in $L = \mathbb{P}^2$ satisfies $C_D D = L$ (well known but see remark below). The result follows since the following inclusion holds $C_{(H \cap L)}(H \cap L) \subset (C_H H) \cap L$ and hence $C_H H$ contains the general 2-plane.

□

Remark. *Let $x \in H$ be a general point and $\mathbb{C}^N \subset P^N$ be an affine chart containing x, where w.l.o.g. $x = 0$. Let the hypersurface $H \cap \mathbb{C}^N$ be given by $f = 0$. The quadractic part f_2 of the Taylor expansion of f at x can not be trivial on $T_x H$, otherwise there would be an open subset of H on which the second fundamental form of H is trivial which would force H to be an hyperplane. Denote by Q_x the quadric defined by $f_2|_{T_x H}$. In the case $H \subset \mathbb{P}^2$ of degree $d \geq 3$, then Q_x not being trivial implies that tangent line $l = T_x H$ is such that $(l \cap H)_x = 2$ and thus l must meet H away from x. Hence $C_H H = Tan(H)$.*

The first appearance of the t-trisecant variety is on the result about the existence of twisted symmetric differentials on smooth hypersurfaces in $\mathbb{P}^N$.

Theorem B. *Let X be a smooth projective hypersurface in $\mathbb{P}^N$. Then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

if and only if the $C_X X = \mathbb{P}^N$, i.e. X is a not quadric.

Proof. Let $\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ be such that it induces a nontrivial $\omega \in H^0(X, S^m[\Omega^1_X(1)])$. As before denote by Ω also the corresponding homogeneous polynomial of degree m in $z_0, ..., z_N$. Proposition 1.5 says that the zero locus $Z(\Omega)$ is such that $\forall x \in X$ $Z(\Omega) \cap T_x X$ is a cone with vertex x. This clearly implies that $X \subset Z(\Omega)$. Moreover since for $x \in X$ $Z(\Omega) \cap T_x X$ is a cone with vertex x containing $X \cap T_x X$, then $C_x X \subset Z(\Omega) \cap T_x X$. Hence:

$$C_X X \subset Z(\Omega) \tag{1.2.8}$$

Therefore if $C_X X = \mathbb{P}^N$ then $\Omega = 0$ and the $(\Leftarrow)$ part of the theorem is proved.

If $C_X X \neq \mathbb{P}^N$ then $X \cap T_x X$ is a cone with vertex x for the general $x \in X$. To see this notice that if $X \cap T_x X$ is not a cone with vertex x then by dimensional arguments $C_x X = T_x X$. Since $Tan(X) = \mathbb{P}^N$ for hypersurfaces, $C_X X \neq \mathbb{P}^N$ implies that $C_x X \neq T_x X$ for the general $x \in X$. The condition that $X \cap T_x X$ is a cone with vertex x is a closed condition on $x \in X$, hence $C_X X = X$ if $C_X X \neq \mathbb{P}^N$.

If $C_X X = X$ then any differential $\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ whose zero locus is a multiple of the hypersurface X will induce a nontrivial element $\omega \in H^0(X, S^m[\Omega^1_X(1)])$. Proposition 1.7 states that if $C_X X = X$ if and only if X is a quadric and finishes the proof. $\square$

Remark. Corollary 1.5 gives that for a smooth quadric $Q \in \mathbb{P}^N$, $H^0(Q, S^m[\Omega^1_Q(1)]) = \mathbb{C}$ if m is even and $H^0(Q, S^m[\Omega^1_Q(1)]) = 0$ if m is odd. There is another way to obtain this result for the smooth quadric $X \subset \mathbb{P}^3$. Then the surface X is $\mathbb{P}^1 \times \mathbb{P}^1$, $\Omega^1_X = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-2, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)$ and $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. Hence $S^m[\Omega^1_X(1)] = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-m, m) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-m+2, m-2) \oplus ... \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(m, -m)$ which gives $H^0(X, S^2[\Omega^1_X(1)]) = \mathbb{C}$ if m is even and $= 0$ if m is odd.

We proceed to analyse the higher codimension case for $X \subset \mathbb{P}^N$. In the hypersurface case the knowledge of the t-trisecant variety X was sufficient to obtain the complete answer in theorem B. But in higher codimension, one should also consider iterations of the construction $C_X Y$. Define $C^2_X Y = C_X(C_X Y)$ (note $C^2_X Y \neq C_{(C_X Y)}(C_X Y)$) and proceed inductively to obtain $C^k_X Y$.

Theorem C. *Let X be a non degenerated smooth projective subvariety of $\mathbb{P}^N$ of dimension $n > 2/3(N-1)$. If $C^k_X X = \mathbb{P}^N$ for some k, Then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

Proof. It follows from the proposition 1.5 that in the dimensional range $n > 2/3(N-1)$ the differentials $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ are induced from symmetric m-differentials $\Omega \in S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ on $\mathbb{C}^{N+1}$. Moreover, Proposition 1.5 also says that the zero locus $Z(\Omega)$ must be such that $Z(\Omega) \cap T_xX$ is a cone with vertex x, $\forall x \in X$. In the proof of theorem B, it was shown that this implies that $C_XX \subset Z(\Omega)$.

Following the same reasoning, since $Z(\Omega) \cap T_xX$ is a cone with vertex x and $C_XX \subset Z(\Omega)$ then $C^2_XX \subset Z(\Omega)$. Repeating the argument one gets $C^l_XX \subset Z(\Omega)$ for all $l \geq 1$. If $C^k_XX = \mathbb{P}^N$ for some k, then clearly every symmetric differential $\Omega \in S^m[\mathbb{C}dz_0 + ... + \mathbb{C}dz_N]$ inducing $\omega \in H^0(X, S^m[\Omega^1_X(1)])$ must be trivial. $\square$

It follows from theorem B and C that it is important to characterize the subvarieties $X \subset \mathbb{P}^N$ with $\dim X > 2/3(N-1)$ with $C^k_XX \neq \mathbb{P}^N$ for any k. They must be special and, as in the hypersurface case, subjectable to a description. A general result showing again the role of quadrics is:

Proposition 1.8. *Let X be a subvariety of $\mathbb{P}^N$ such that $X \subset Q_1 \cap ... \cap Q_l$, where Q_1,..., Q_l are quadrics. Then $C^k_XX \subset Q_1 \cap ... \cap Q_l$ for all $k \geq 1$.*

Proof. There is the inclusion of the t-trisecant varieties $C_XX \subset C_{Q_i}Q_i$ for all quadrics Q_i $i = 1, ..., l$. The equality $C_{Q_i}Q_i = Q_i$ proved in proposition 1.6 gives $C_XX \subset Q_1 \cap ... \cap Q_l$. In an equal fashion one sees that $C^2_XX = C_X(C_XX) \subset C_{Q_i}Q_i$ for all $i = 1, ..., l$. Induction then gives the result. $\square$

Remark. *One should investigate the conditions on X for the validity of the assertion that C^k_XX is the intersection of all quadrics containing X for k sufficiently large.*

The case where X is of codimension 2, $X^n \subset \mathbb{P}^{n+2}$, also has a complete answer, see proposition 1.12 below. The answer it will follow from establishing that C_XX is the trisecant of X, $Tr(X)$, if $n \geq 3$, then one can use the results on trisecant varieties of varieties of codimension 2 of Ziv Ran [Ra83], $n \geq 4$, and Kwak [Kw02] for the threefold case.

Let $X \subset \mathbb{P}^N$ be a subvariety and $l \subset \mathbb{P}^N$ a line meeting X at k points, x_i $i = 1, ..., k$, the line l is said to is of type $(n_1, ..., n_k)$ if $n_i = length_{x_i}(X \cap l)$. A line l is a trisecant line if $\sum n_i \geq 3$ and a t-trisecant line if additionally one of the $n_i \geq 2$ ($Tr(X)$ is the union of all trisecant lines and C_XX is the union of all t-trisecant lines).

Lemma 1.9. *Let X be a subvariety of $\mathbb{P}^N$ and $\pi : L \to T$ a 1-dimensional family of lines in $\mathbb{P}^N$ all passing through a fixed $z \notin X$ and whose union is not a line. If the general line meets X at least twice, then one of the lines must meet X with multiplicity at least 2 at some point.*

Proof. Let H be an hyperplane not containing z and $f : T \to H$ be the map which sends t to $L_t \cap H$. Denote by C the image of map f. Let $C(z, C)$ the cone over C with vertex

z. Let D be the curve which consists of the divisorial component of $X \cap C(z, C)$. The possibly nonreduced curve D is such that any line l_c joining z to $c \in C$ meets D at least twice (counting with multiplicity). We can assume that the lines l_c meet X with at most multiplicity at any $x \in X$, otherwise the result follows. Hence the the curve $D \subset C(z, C)$ is reduced and clearly does not pass through z.

Resolve the cone $C(z, C)$ by normalizing C, $\bar{C}$, and blowing up the singularity at the vertex. The resulting surface Y is a ruled surface over $\bar{C}$, which comes with two maps $\sigma : Y \to C(z, C)$ and $f : Y \to \bar{C}$. Let $\bar{D}$ be the pre-image of D by σ. If $\bar{D}$ meets any of the fibers of $f : Y \to \bar{C}$ with multiplicity ≥ 2 then as before we are done. Hence $\bar{D}$ is smooth moreover it must be a multi-section. This is impossible since by base change we would obtain a ruled surface which would have at least two disjoint positive sections not intersecting (the unique negative section lies over the the pre-image of p). $\square$

We proceed by giving an alternative proof of Zak's theorem on the equality of the secant and tangent variety for smooth subvarieties X whose secant variety does not have the expected dimension.

Corollary 1.10. *(Zak's Theorem) Let X be a smooth subvariety of $\mathbb{P}^N$. If $\dim Sec(X) < 2n+1$ then $Tan(X) = Sec(X)$.*

Proof. Assume $Sec(X) \neq X$ since if the equality holds then clearly $Tan(X) = Sec(X)$. Let z be a point of $Sec(X) \setminus X$. Since $Sec(X)$ has less than the expected dimension there is a positive dimensional family $\pi : L \to T$ of secant lines passing through z. Apply lemma 1.10 to a 1-dimensional subfamily of $\pi : L \to T$ and obtain that one of this lines L_{t_0} must meet X with multiplicity at least 2 at some point $x \in X$ hence L_{t_0} is tangent to X at x and $z \in Tan(X)$. $\square$

Finally we use the lemma to describe an important case when the trisecant variety is equal to the t-trisecant variety.

Corollary 1.11. *Let X be a smooth subvariety of $\mathbb{P}^N$. If the family of trisecant lines of X through a general point of $Tr(X)$ is at least 1-dimensional, then $TrX = C_X X$.*

Proof. The same argument after replacing $Sec(X)$ by $Tr(X)$ and $Tan(X)$ by $C_X X$. $\square$

Proposition 1.12. *Let X be a smooth subvariety of codimension 2 in $\mathbb{P}^{n+2}$. If $n \geq 3$ then:*

1) $C_X X = Tr(X)$.

2) $C_X X = \mathbb{P}^{n+2}$ or $C_X X$ is the intersection of the quadrics containing X.

Proof. First we establish 1). Let z be a general point of the trisecant variety $Tr(X)$. Let l be a trisecant line passing through z, assume it is not also t-trisecant since otherwise there is nothing to prove. Consider the projection $p_z : X \to \mathbb{P}^{n+1}$ from the point z to an hyperplane $\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$. Denote 3 of the points in $l \cap X$ by x_1, x_2 and x_3 and $p = p_z(x_i) = l \cap \mathbb{P}^{n+1}$. The hypersurface $p_z(X) \subset \mathbb{P}^{n+1}$ has at p a decomposition into local irreducible components $p_z(X) \cap U_p = H_1 \cup ... \cup H_k$, where U_p is a sufficiently small neighborhood of p. The points x_i $i = 1, ..., 3$ have neighborhoods U_i such that $p_z : U_i \to p_z(U_i)$ is finite and $p_z(U_i)$ contains one of H_j. Consider the case where the local irreducible components H_j contained by $p_z(U_i)$ are all distinct, w.l.o.g. denote them by H_1, H_2 and H_3 (the other cases will follow by the same argument and are more favourable to our purposes). In this case $H_1 \cap H_2 \cap H_3$ will be of dimension $n-2$. Since for every point $t \in H_1 \cap H_2 \cap H_3$ the line $\overline{zt}$ is trisecant, the result follows from corollary 1.11.

The part 2) follows from known facts about the trisecant varieties of smooth varieties X of codimension 2 in projective space $\mathbb{P}^N$. The trisecant variety is irreducible if the dimension of X $n \geq 2$. Ziv Ran [Ra83] showed that if $n \geq 4$ and $Tr(X) \neq \mathbb{P}^{n+2}$ then X must be contained in a quadric (this result is not explicitly stated but clearly follows from the article). Later Kwak [Ka02] showed that the same holds for $n = 3$. Ran also showed that if the degree of X is less or equal to its dimension then X, $d \leq n$, then X is a complete intersection.

We assume X is nondegenerate in $\mathbb{P}^N$ (the degenerate case follows the from hypersurface case). The above paragraph implies that if $\dim Tr(X) = n+1$ then $C_X X = Tr(X)$ is the quadric containing X. The case $\dim Tr(X) = n$ or equivalently $Tr(X) = X$ is settled by a slicing argument and the case $n = 3$. It is known, see for example remark 3.6 of [Ka02], that if X is of dimension 3 and $Tr(X) = X$ then X is a complete intersection of two quadrics or the Segree variety $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ which is the intersection of 3 quadrics. If $n \geq 4$ consider a general 5-plane $L \subset \mathbb{P}^N$, then $X \cap L$ is a smooth 3-fold in $L = \mathbb{P}^5$ for which $Tr(X \cap L) = X \cap L$, since $Tr(X \cap L) \subset Tr(X) \cap L$ and $X \cap L \subset Tr(X \cap L)$. Then $X \cap L$ is one of the two cases described above. Both cases have degree equal to 4 hence the degree X is also 4. It follows from the result of Ran and the end of the previous paragraph that X is a complete intersection of two quadrics. $\square$

Theorem D. *Let X be a smooth subvariety of codimension 2 in $\mathbb{P}^{n+2}$. If $n \geq 3$ then:*

$$H^0(X, S^m[\Omega^1_X(1)]) = 0$$

if and only if X is not contained in a quadric.

Proof. If X is not contained in a quadric, then $C_X X = \mathbb{P}^{n+2}$ by proposition 1.12. The vanishing $H^0(X, S^m[\Omega^1_X(1)]) = 0$ follows theorem C.

To analyse the case where X is contained in a quadric Q recall that proposition 1.5 states that $H^0(X, S^m[\Omega^1_X(1)]) = \{\Omega \in S^m[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N] |\ Z(\Omega|) \cap T_x X)$ is a cone with vertex at x, $\forall x \in X\}$. Consider the symmetric differential $\Omega_Q \in S^2[\mathbb{C}dz_0 \oplus ... \oplus \mathbb{C}dz_N]$ associated with the quadric Q. For all $x \in X$ $Z(\Omega_Q) \cap T_x X) (= Q \cap T_x X)$ is a cone with vertex x since $T_x X \in T_x Q$. Hence Ω_Q defines an element of $H^0(X, S^m[\Omega^1_X(1)])$ and this element is nontrivial since $Tan(X) = \mathbb{P}^{n+2}$. $\square$

1.3 Symmetric differentials on subvarieties of abelian varieties.

In this section we do a short presentation of the results which are the analogue to theorem A and part of theorem C for subvarieties of abelian varieties. Again we are having in mind subvarieties with "low" codimension. Recently, Debarre [De06] using the same perspective tackled the problem of which subvarieties have an ample cotangent bundle, which are in the other end in terms of codimension.

Let X be a smooth subvariety of an abelian variety A^n. The surjection on the conormal exact sequence:

$$0 \to N^*_{X/A^n} \to \Omega^1_{A^n}|_X \to \Omega^1_X \to 0$$

induces the inclusion $j : \mathbb{P}(\Omega^1_X) \to \mathbb{P}(\Omega^1_{A^n}|_X)$ of projectivized cotangent bundles. The projectivized cotangent bundle of A^n is trivial, i.e. $\mathbb{P}(\Omega^1_{A^n}) \simeq A^n \times \mathbb{P}^{n-1}$. Let $p_2 : \mathbb{P}(\Omega^1_{A^n}) \to \mathbb{P}^{n-1}$ denote the projection onto the second factor. Then $\mathcal{O}_{\mathbb{P}(\Omega^1_{A^n})}(m) \simeq p_2^*\mathcal{O}_{\mathbb{P}^{n-1}}(m)$. The composed map $f = p_2 \circ j$:

$$f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$$

is called the tangent map for X in A^n.

Theorem F. *Let X be a smooth subvariety of an abelian variety A^n. If the tangent map $f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$ is both surjective and connected then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X) = H^0(A^n, S^m\Omega^1_{A^n})$$

.

Proof. Associated to the tangent map f is the isomorphism:

$$\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(m) \simeq f^*\mathcal{O}_{\mathbb{P}^{n-1}}(m) \tag{1.2.9}$$

As in the proof of lemma 1, the isomorphism (1.2.9) and the connectedness and surjectivity of the tangent map f give that:

$$H^0(\mathbb{P}(\Omega^1_X), f^*(\mathcal{O}_{\mathbb{P}^{n-1}}(m))) = f^*H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m))$$

The result follows from the identifications $H^0(X, S^m\Omega^1_X) = H^0(\mathbb{P}(\Omega^1_X), \mathcal{O}_{\mathbb{P}(\Omega^1_X)}(m))$ and $H^0(A^n, S^m\Omega^1_{A^n}) = p_2^*H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(m))$. □

Corollary 1.13. *Let X be a smooth hypersurface of an abelian variety A^n with $n > 2$ which does not contain any translate of an abelian subvariety of A^n. Then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X) = H^0(A^n, S^m\Omega^1_{A^n})$$

Proof. It follows from theorem F that it is enough to show that the tangent map f is connected and surjective. The hypothesis on X implies that X itself is not the translate of an abelian subvariety and hence the tangent map is surjective.

The tangent map $f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$ induces the map $\gamma : X \to \mathbb{G}(n-1, n)$, which is the Gauss map for X in A^n. The fibers $f^{-1}(p)$ for $p \in \mathbb{P}^{n-1}$ project to $F_p = \pi(f^{-1}(p)) \subset X$. The set F_p consists of all the points $x \in X$ for which the line in T_0A^n corresponding to p is contained in T_xX. We are using the common identification of the tangent space T_xA^n for any $x \in A^n$ with $T_0A^n = \mathbb{C}^n$, which sends the tangent spaces to X at $x \in X$ to an hyperplane of T_0A^n. The Grassmanian $\mathbb{G}(n-1, n)$ is $\mathbb{P}^{n-1}$ and the subvariety $W \subset \mathbb{G}(n-1, n)$ consisting of all $n-2$-planes passing through the point $p \in \mathbb{P}^{n-1}$ is a hyperplane $H \subset \mathbb{G}(n-1, n)$.

The hypothesis on X guarantee the the Gauss map is finite (see [Ab94] or see corollary 3.10 of [Za93]). Hence the image of X under the Gauss map is at least of dimension 2. It follows then that $F_p = \gamma^{-1}(\gamma(X) \cap H)$ is connected by Bertini's theorem and hence $f^{-1}(p)$ is also connected since $\pi : f^{-1}(p) \to F_p$ is 1 to 1. □

As in the case of subvarieties of $\mathbb{P}^N$ we obtain a vanishing theorem.

Theorem G. *Let X be a smooth subvariety of an abelian variety A^n. If the general fiber of the tangent map $f : \mathbb{P}(\Omega^1_X) \to \mathbb{P}^{n-1}$ is positive dimensional then $\forall m \geq 0$:*

$$H^0(X, S^m\Omega^1_X \otimes L) = 0$$

if L is a negative line bundle on X.

Proof. It follows from the arguments of theorem F and theorem A. □

2. The non-invariance of the cotangent plurigenera

Let X be a smooth projective variety. As in [Sa78], we define:

$$Q_m(X) = \dim H^0(X, S^m\Omega^1_X) \tag{2.1.1}$$

The dimension $Q_m(X)$ is called the symmetric m-genus of X. The graded ring $\Omega(X) = \sum_{m=0}^{\infty} H^0(X, S^m\Omega^1_X)$ is called the cotangent ring of X. We define the cotangent dimension of X to be:

$$\lambda_I(X) = \dim_{Iitaka} \mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1) \tag{2.1.2}$$

where $\dim_{Iitaka} \mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ is the Iitaka dimension of the the line bundle $\mathcal{O}_{\mathbb{P}(\Omega^1_X)}(1)$ on $\mathbb{P}(\Omega^1_X)$. For example, it follows from the results of the previous section that if X is a smooth subvariety of $\mathbb{P}^N$ with $\dim_{\mathbb{C}} X > N/2$ then $\lambda_I(X) = -\infty$. An abelian variety X of dimension n has $\lambda_I(X) = n-1$ and a smooth variety Y of dimension n with ample cotangent bundle has the maximal possible Iitaka cotangent dimension for varieties of dimension n, $\lambda_I(Y) = 2n-1$.

The symmetric 1-genus, $Q_1(X)$, of a smooth projective variety X is also called the irregularity of X. The irregularity of a Kahler manifold is a topological invariant (as follows from Hodge theory) and hence it can not jump in smooth families. One can also see the irregularity of X as one of the plurigenus of X, more precisely $P_1(X)$ where $P_m(X) = \dim H^0(X, (\bigwedge^n \Omega^1_X)^{\otimes m})$. There is an amazing result of Siu [Si98] that states that all plurigenera are invariant in smooth families of projective varieties.

The symmetric plurigenera behaves differently. The first author gave the first example of a smooth family of projective varieties where the cotangent m-genus jumps [Bo78]. We start by presenting self contained modification of that example.

Let $A^3 = \mathbb{C}^3/\Lambda$ be an abelian 3-fold, where z_1, z_2 and z_3 are the Euclidean holomorphic coordinates of $\mathbb{C}^3$. We denote the involution of T^3 given by the map $z \to -z$ in $\mathbb{C}^3$ by $\sigma : T^3 \to T^3$. Let X_t be a one-dimensional family, over the disc Δ, of σ-invariant smooth hypersurfaces of T^3 which pass through only one of the fixed points p_0 of the involution σ. Moreover, locally on a neighborhood of the fixed point p_o, which we assume to be $p_0 = (0,0,0)$, X_t is given by the equation:

$$z_1 = tz_2^3 + f_t(z_2, z_3) \tag{2.1.3}$$

where $f_t(z_2, z_3) \in (z_2, z_3)^5$.

Theorem H. *Let X_t be the family described in (2.1.3) and Y_t be the family which is the simultaneous minimal resolution of the family nodal varieties $V_t = X_t/\sigma$. Then:*

a) The symmetric plurigenera is not invariant along the family Y_t.

b) Under natural identifications the following holds:

$$H^0(Y_0, S^m\Omega^1_{Y_0}) \supset [\bigoplus_{3m_1 \geq m_2+m_3} \mathbb{C}dz_1^{m_1}dz_2^{m_2}dz_3^{m_3}]^{\mathbb{Z}_2}$$

$$H^0(Y_t, S^m\Omega^1_{Y_t}) = [\bigoplus_{m_1 \geq m_2+m_3} \mathbb{C}dz_1^{m_1}dz_2^{m_2}dz_3^{m_3}]^{\mathbb{Z}_2}, \ t \neq 0$$

Proof. Let us set up some notation. The maps $q_t : X_t \to X_t/\sigma$ and $r_t : Y_t \to X_t/\sigma$ denote respectively the quotient map induced by σ and the minimal resolution of X_t/σ. Let $\hat{X}_t$ be the family whose members are the blow up of X_t at the fixed point p_0. There are two maps from $\hat{X}_t$ to consider, one is the blow up map $b_t : \hat{X}_t \to X_t$ and the other is the double cover $g_t : \hat{X}_t \to Y_t$ ramified at the -2-curve of Y_t. The blow up b_t induces an isomorphism $(db_t)^* : H^0(X_t, S^m\Omega^1_{X_t}) \to H^0(\hat{X}_t, S^m\Omega^1_{\hat{X}_t})$ and the double cover g_t induces an inclusion $(dg)^* : H^0(Y_t, S^m\Omega^1_{Y_t}) \hookrightarrow H^0(\hat{X}_t, S^m\Omega^1_{\hat{X}_t})$. Using the identifications induced by these maps, one has the injection:

$$j_t : H^0(Y_t, S^m\Omega^1_{Y_t}) \hookrightarrow H^0(X_t, S^m\Omega^1_{X_t})$$

The next goal is to figure out what are the conditions required on $w \in H^0(X_t, S^m\Omega^1_{X_t})$ for w to belong to $H^0(Y_t, S^m\Omega^1_{Y_t})$. It follows from corollary 1.13 that the m-the order symmetric differentials on X_t are the restrictions to X_t of the m-symmetric differentials ω of the abelian variety T^3 (T^3 can be chosen to be simple so that the hypothesis of the corollary holds for X):

$$H^0(X_t, S^m\Omega^1_{X_t}) = (di_t)^*\mathrm{S}^m[\mathbb{C}dz_1 + \mathbb{C}dz_2 + \mathbb{C}dz_3]$$

where $i_t : X_t \hookrightarrow A^3$ are the inclusion maps. The goal now is to find which symmetric differential monomials $dz_1^{m_1}dz_2^{m_2}dz_3^{m_3}$ on A^3 induce symmetric differentials on X_t which are also symmetric differentials on Y_t. To accomplish this task we describe the local picture for the symmetric differentials around the fixed point p_0. The hypersurfaces X_t are given locally around $p_0 = (0,0,0)$ by $z_1 = tz_2^3 + f_t(z_2, z_3)$ with $f_t(z_2, z_3) \in (z_2, z_3)^5$, so there is a neighborhood $U_t \subset X_t$ of p_0 with (z_2, z_3) as local coordinates. The symmetric differential monomial $dz_1^{m_1}dz_2^{m_2}dz_3^{m_3}$ on A^3 induces the following symmetric differential on U_t:

$$(di_t)^*(dz_1^{m_1}dz_2^{m_2}dz_3^{m_3})|_{U_t} = ct^{m_1}z_2^{2m_1}dz_2^{m_1+m_2}dz_3^{m_3} + \gamma_t \tag{2.1.4}$$

where the $(m_1+m_2+m_3)$-symmetric differential γ_t has the coefficients of their monomial terms in $(z_2, z_3)^{2m_1+2}$. If $t = 0$ we have $(di_0)^*(dz_1^{m_1}dz_2^{m_2}dz_3^{m_3})|_{U_0}$ a $(m_1 + m_2 + m_3)$-symmetric differential whose coefficients of their monomial terms belong to $(z_2, z_3)^{4m_1}$.

The lemma below examines the local question of which symmetric differentials on U_t corresponds to symmetric differentials on a neighborhood of -2-curve in V_t. Let $(U, 0)$ be the neighborhood germ of the origin in $\mathbb{C}^2$ with complex coordinates (u_1, u_2) and $(U/\sigma, x_0)$ be the neighborhood germ of the nodal surface singularity (σ is again the -id involution). Let (V, E) be the neighborhood germ of the (-2)-curve and $r : (V, E) \to (U/\sigma, x_0)$ the minimal resolution.

Lemma 2.1. *A symmetric differential monomial $w \in H^0(U, S^m\Omega^1_U)$ of the form $\omega = h(u_1, u_2)du_1^{m_1}du_2^{m_2}$ induces a symmetric differential in $H^0(V, S^m\Omega^1_V)$ if and only if w is σ-invariant and $h(u_1, u_2) \in (u_1, u_2)^{m_1+m_2} \subset \mathcal{O}(U)$.*

Proof. Let (W, E') be the germ neighborhood of the blow up of $(U, 0)$ at 0 and $b : (W, E') \to (U, 0)$ the blow up map. Denote by $g : (W, E') \to (V, E)$ be the 2 to 1 naturally defined covering of V ramified at $E \subset V$ (for which $\sigma \circ b = r \circ g$ holds).

First, we note that there is a natural bijection between $H^0(V \setminus E, S^m\Omega^1_V)$ with $[H^0(W \setminus E', S^m\Omega^1_W)]^{\mathbb{Z}_2}$. The differential pullback $dg^* : H^0(V, S^m\Omega^1_V) \to H^0(W, S^m\Omega^1_W)$

is an injection. We want to see the differentials of W on U, to do this we notice that there is a natural identification $H^0(U, S^m\Omega^1_U) = H^0(W, S^m\Omega^1_W)$. Hence, we have that $H^0(V, S^m\Omega^1_V) \subset [H^0(U, S^m\Omega^1_U)]^{\mathbb{Z}_2}$. What remains to be determined is what to require on the elements of $[H^0(U, S^m\Omega^1_U)]^{\mathbb{Z}_2}$ for them to be, after the natural identification mentioned above, pullbacks by dg^* .

We give a coordinate chart approach to this problem. Consider the single affine chart blow up map $f : (\mathbb{C}^2, u, v) \to (\mathbb{C}^2, u_1, u_2)$ given by $(u, v) \to (u, uv)$. Consider also the double ramified covering $g : (\mathbb{C}^2, u, v) \to (\mathbb{C}^2, x, y)$ given by $(u, v) \to (u^2, v)$.

The relations between the differentials in the different coordinate charts $\mathbb{C}^2$ are:

$$f^*(du_1) = du, \quad f^*(du_2) = udv + vdu \quad ; \quad g^*(dx) = 2udu, \quad g^*(dy) = dv \tag{2.1.5}$$

Let us write the pullback by f of a symmetric differential on $(\mathbb{C}^2, u_1, u_2)$:

$$f^*(du_1^{m_1} du_2^{m_2}) = \sum_{i=0}^{m_2} \binom{m_2}{i} u^{m_2-i} v^i dv^{m_2-i} du^{m_1+i} \tag{2.1.6}$$

From (2.1.4) one easily sees that the pullback by f of any symmetric differential monomial of order m $(m_1 + m_2 = m)$ has the term in du^m with no power of u in the coefficient (and all other terms have du with an order smaller than m). On the other hand, by (2.1.5) a symmetric differential $\omega = u^{i_1} v^{i_2} du^{m_1} dv^{m_2}$ is a pullback by dg^* if and only if $i_1 \geq m_1$ and $i_1 \equiv m_1 \mod (2)$. This plus the coordinate description of f implies that $f^*(h(u_1, u_2) du_1^{m_1} du_2^{m_2})$ with $m_1 + m_2 = m$ is a pullback by g only if the Taylor expansion of h has all terms with combined order in u_1 and u_2 to be greater or equal to m, which concludes the proof.

□

Applying lemma 2.1 to (2.1.4) gives that symmetric differentials on X_t induced by the symmetric differential monomials $dz_1^{m_1} dz_2^{m_2} dz_3^{m_3}$ on A^3 are also symmetric differentials on Y_t:

(for $t \neq 0$) if and only if and $m_1 \geq m_2 + m_3$ and $m_1 + m_2 + m_3 \equiv 0 \mod 2$.

(for $t = 0$) if and only if and $3m_1 \geq m_2 + m_3$ and $m_1 + m_2 + m_3 \equiv 0 \mod 2$.

The above implies part b) of the theorem and concludes the proof.

□

Remark. The results in the proof of theorem H also give the following geometric mechanism to create jumping of the symmetric plurigenera. Consider a family of σ-invariant hypersurfaces X_t in A^3 passing through two fixed points $p_0 = (a_0, b_0, c_0)$ and $p_1 = (a_1, b_1, c_1)$ of σ. Locally at the fixed points p_0 and p_1 X_t is of the form $(z_1')_t - a_i = (z_2 - b_i)^3 + f_i(z_2 - b_i, z_3 - c_i)$, $i = 0, 1$, where $(z_1')_t$ is a linear expression on z_1, z_2 and z_3 and $f_i(z_2 - b_i, z_3 - c_i)$ vanishes at (b_i, c_i) with order larger than 5. Built $(z_1')_t$ such that for $t \neq 0$ the tangent spaces to X_t at p_0 and p_1 do not coincide, after using the global identification of tangent spaces on the abelian variety A^3, but at $t = 0$ they do coincide. For such a family we would have by the same arguments of the theorem that $H^0(Y_t, S^m\Omega^1_{Y_t}) = 0$ for $t \neq 0$ and $H^0(Y_0, S^m\Omega^1_{Y_0}) = [\bigoplus_{m_1 \geq m_2+m_3} \mathbb{C}dz_1^{m_1}dz_2^{m_2}dz_3^{m_3}]^{\mathbb{Z}_2}$, Y_t is again the family which is the simultaneous minimal resolution of the family nodal varieties $V_t = X_t/\sigma$.

We now answer a question posed by Paun: are the dimensions of $H^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ constant for a family of smooth projective varieties? We answer this question negatively.

Theorem I. *Let Y_t be a family of smooth projective varieties. The invariance of the dimension of $H^0(Y_t, S^m\Omega^1_{Y_t} \otimes K_{Y_t})$ does not necessarily hold along the family.*

Proof. Let X_t be a family over Δ of smooth hypersurfaces of degree d of $\mathbb{P}^3$ specializing to a nodal hypersurface X_0 with $l > \frac{8}{3}(d^2 - \frac{5}{2}d)$ nodes. This is possible as long $d \geq 6$ [Mi83]. Denote by Y_t the family which is the simultaneous resolution of the family X_t, $t \in \Delta$. The general member of Y_t is a smooth hypersurface of $\mathbb{P}^3$ of degree d and Y_0 is the minimal resolution of X_0. We proved in [BoDeO06] that if $d \geq 6$ Y_0 has plenty of symmetric differentials, more precisely $H^0(Y_0, S^m\Omega^1_{Y_0}) \uparrow m^3$. This result plus the effectivity of the canonical divisor K_{Y_0}, in particular, implies that:

$$H^0(Y_0, S^m\Omega^1_{Y_0} \otimes K_{Y_0}) \neq 0, \quad m \gg 0 \tag{2.1.7}$$

Theorem B gives that $H^0(Y_t, S^m\Omega^1_{Y_t} \otimes \mathcal{O}_{Y_t}(m)) = 0$. The canonical divisor of the hypersurface Y_t is $K_{Y_t} = \mathcal{O}_{Y_t}(d-4)$, which implies that :

$$H^0(Y_t, S^m\Omega^1_{Y_t} \otimes K_{Y_t}) = 0, \quad m \geq d-4 \tag{2.1.8}$$

The result follows from (1.2.7) and (1.2.8). ☐

We want to modify the question of Paun to be able to express a stronger result (which is suggested by the proof of the above theorem). We introduce the following notation:

$$Q_{\alpha,m}(X) = \dim H^0(X, S^m(\Omega^1_X \otimes \alpha K_X)) \tag{2.1.9}$$

The dimension $Q_{\alpha,m}(X)$ is called the α-twisted symmetric m-genus of X. An interesting question to ask is: what is the lower bound β for the$m\alpha$'s for which the α-twisted plurigenera is invariant along all families of smooth projective varieties X_t with $K_{X_t} > 0$?

Corollary 2.2. *The lower bound β for the α's for which the α-twisted plurigenera is invariant for the families of smooth projective surfaces X_t with $K_{X_t} > 0$ must satisfy:*

$$\beta \geq 1/2$$

Proof. Consider the families Y_t described in theorem 1 for degree $d = 6$. In this case $K_{Y_t} = 2\mathcal{O}_{Y_t}(1)$ and by theorem B we have $H^0(X_t, S^m(\Omega^1_{Y_t} \otimes 1/2K_{Y_t})) = 0$ for $t \neq 0$. On the other hand, $H^0(Y_0, S^m(\Omega^1_{Y_0} \otimes 1/2K_{Y_0})) \neq 0$ for $m \geq 2$ as in theorem I. □

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