TWISTED SYMMETRIC DIFFERENTIALS AND THE QUADRIC ALGEBRA OF SUBVARIETIES OF $\mathbb{P}^N$ OF LOW CODIMENSION

BRUNO DE OLIVEIRA AND CHRISTOPHER LANGDON

ABSTRACT. A smooth subvariety $X \subset \mathbb{P}^N$ has for each $\alpha \in \mathbb{Q}$ an associated algebra $Q(X,\alpha) = \bigoplus_{m\alpha\in\mathbb{Z}} H^0(X, S^m[\Omega^1_X(\alpha)])$. The algebra $Q(X,0)$ is the the intrinsic algebra of symmetric differentials and $Q(X,1)$ is called the algebra of twisted symmetric differentials. In this paper we show that when X is a complete intersection of dimension dim $X \geq \max\{2/3(N-1), 3\}$, then the algebra of twisted symmetric differentials is the quadric algebra of X, i.e. $Q(X,1) \simeq Sym(H^0(\mathbb{P}^N, \mathcal{I}_X(2)))$. We establish an identification of the twisted symmetric m-differentials on X with the tangentially homogeneous polynomials relative to X of degree m. We also obtain the same isomorphism if X is codimension two and $\dim X \geq 3$. The lack of the hypothesis of X being a complete intersection is counter-balanced by properties of the vanishing locus of tangentially homogeneous polynomials and algebraic geometric properties of the tangent-secant variety of X. Symmetric differentials and Quadrics and Trisecant variety and Low codimension 14M07 and 14N99 and 14M10

1. Introduction

A smooth subvariety $X \subset \mathbb{P}^N$ has for each $\alpha \in \mathbb{Q}$ an associated algebra $Q(X,\alpha) = \bigoplus_{m\alpha\in\mathbb{Z}} H^0(X, S^m[\Omega^1_X(\alpha)])$. The case $\alpha = 0$ corresponds to the intrinsic algebra of symmetric differentials on X and plays a significant role in classification problems and questions concerning the existence of subvarieties in X that are not of general type, [1], [6], [2], [5], [4]. The special algebra corresponding to $\alpha = 1$ is called the algebra of twisted symmetric differentials. The algebra $Q(X,1)$ is an extrinsic algebra that encodes clear geometric properties about the embedding which in turn gives information on X and $Q(X,0)$. In this paper we study the geometry encoded by the algebra of twisted symmetric differentials when $X \subset \mathbb{P}^N$ is of low codimension with an emphasis on its connection to the quadric algebra of X, $QA(X) \subset \mathbb{C}[X_0, ..., X_N]$, which is generated by the quadratic polynomials vanishing on X.

Our motivation for studying the algebra $Q(X,1)$ comes from a few different directions. The algebra $Q(X,1)$ can be viewed as an extremal/boundary case among all the algebras $Q(X,\alpha)$, $\alpha \in \mathbb{Q}$, due to a vanishing result of M. Schneider [15] stating that $Q(X,\alpha) = 0$ for all $X \subset \mathbb{P}^N$ with $\dim X > N/2$ if and only if $\alpha < 1$. Also, $Q(X,1)$ plays a role in the study of varieties with ample cotangent bundle as seen in the work of O. Debarre [6] and it was used by Bogomolov and the first author [3] to show that the dimension $h^0(X_t, S^m\Omega^1_{X_t} \otimes K_{X_t})$ is not necessarily invariant in families even if K_{X_t} is big for all t (answering a question of M. Paun).

The extremal/boundary behavior of the algebra $Q(X,1)$ among the algebras $Q(X,\alpha)$, indicates that interesting geometry should be involved in determining

$Q(X,1)$. Using the Euler and co-normal sequences for $\mathbb{P}^N$, X and the affine cone $\tilde{X} \subset \mathbb{C}^{N+1}$ over X, see theorem 2.1, we show how the tangent map $t : \mathbb{T}X \to \mathbb{P}^N$ plays a role in determining $Q(X,1)$ ($\mathbb{T}X$ is the projective bundle of embedded tangent spaces of X in $\mathbb{P}^N$). The projective bundle $\mathbb{T}X$ is realized by $\mathbb{P}(\widetilde{\Omega}_X(1))$, where $\widetilde{\Omega}_X(1)$ is the sheaf of twisted extended differentials on X, i.e the extension of the sheaf of twisted differentials $\Omega^1_X(1)$ induced by the Euler sequence.

In the dimensional range $\dim X > \max\{1, 2/3(N-1)\}$ the tangent map is both surjective and connected [3]. A consequence of this is that sections of $S^m[\widetilde{\Omega}_X(1)]$ naturally come from homogeneous polynomials of degree m in the homogeneous coordinate ring of $\mathbb{P}^N$. The following theorem provides an explicit identification of the algebra of twisted symmetric differentials $Q(X,1)$ with a sub-algebra of $\mathbb{C}[X_0,...,X_N] \simeq \bigoplus_{m=0}^{\infty} H^0(X, S^m[\widetilde{\Omega}_X(1)])$.

Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate smooth subvariety with $n > \max\{1, 2/3(N-1)\}$. Then there is a graded isomorphism of algebras induced by the tangent map:

$$Q(X,1) \simeq \mathbb{C}[X_0,...,X_N]^h_{TX}$$

where $\mathbb{C}[X_0,...,X_N]^h_{TX} \subset \mathbb{C}[X_0,...,X_N]$ is the subalgebra generated by the polynomials that are tangentially homogeneous relative to X.

A geometric way to characterize the homogeneous polynomials that are tangentially homogeneous relative to X is to say that they define at all embedded tangent spaces $\mathbb{T}_x X$ a cone with vertex x. The complete characterization of tangentially homogeneous polynomials relative to $X \subset \mathbb{P}^N$ is attained for smooth complete intersections whose tangent variety $\mathrm{Tan}(X) = \mathbb{P}^N$.

Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate smooth complete intersection in the dimensional range $n > \max\{1, 2/3(N-1)\}$. Then there is a graded isomorphism:

$$Q(X,1) \simeq Sym(H^0(\mathbb{P}^N, \mathcal{I}_X(2)) =: QA(X)$$

In proving theorem 3.1 we used the seminal work of Griffiths and Harris [8], see also [12], on the role of algebraic differential geometry in algebraic geometry. Also useful to us was the work of Russo and Ionescu on the Hartshorne conjecture for varieties defined by quadrics, [10]. More precisely, we used the relations between the properties of the second fundamental form and properties of the tangent map, the dual variety and the Gauss map (the latter two playing a role in later sections).

A non-constant polynomial H that is tangentially homogeneous with respect to X must vanish on X and hence can be written as $H = \sum G_{i_1...i_k} F_1^{i_1}...F_k^{i_k}$ where $G_{i_1...i_k} \notin I(X)$ where the F_i belong to a set $\{F_1,...,F_m\}$ of generators of $I(X)$. Let $f_{i,x}$ be a dehomogenization of F_i (adapted to $T_x X$). Algebraic differential geometry can be used to show that if $\mathrm{Tan}(X) = \mathbb{P}^N$, then the quadratic terms $f^{(2)}_{i,x}$ of the Taylor expansions at x of the $f_{i,x}$ along $T_x X$ are algebraically independent. This result plus the property that H is tangentially homogeneous relative to X implies that the only F_i involved in H are quadratic and the $G_{i_1...i_k} \in \mathbb{C}$.

The above work motivates:

Conjecture 1.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate smooth subvariety with $n > max\{1, 2/3(N-1)\}$. Then there is a graded isomorphism:*

$$Q(X,1) \simeq QA(X)$$

The Hartshorne conjecture states that in the low codimensional range $n > 2/3N$ a smooth subvariety $X^{(n)} \subset \mathbb{P}^N$ must be a complete intersection, [9] (see also [10]). The conjecture 1.1 would be a corollary of theorem 3.1 (for $n > 2/3N$) if Hartshorne's conjecture were to be settled, but the conjecture is still open even for codimension two. Nevertheless, the conjecture 1.1 can be proven for codimension two.

Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate smooth subvariety with $\text{codim}(X) = 2$ and $n \geq 3$. Then conjecture 1.1 holds.

To prove the above theorem without the assumption of X being a complete intersection we needed results on the trisecant varieties associated to X, which are also of independent interest.

The theorem naturally breaks into three cases determined by the quadratic index of X, i.e. $i_Q(X) = h^0(X, \mathcal{I}_X(2))$. The case $i_Q(X) \geq 2$ can be handled since by results of Ziv Ran on Hartshorne's conjecture [13] X is either a complete intersection or the Segre cubic scroll $\Sigma_{1,2} \subset \mathbb{P}^5$ (see also [11]). Hence this case follows from theorem 3.1 and the case $X = \Sigma_{1,2}$ which is more delicate and dealt with explicitly.

Less is known on the Hartshorne conjecture for the other two cases where $i_Q(X) = 0$ and $i_Q(X) = 1$. These cases can be handled by using the observation that a non-constant tangentially homogeneous polynomial relative to X must vanish on the tangent-secant lines to X, i.e. the trisecant lines that are tangent and meet X in at least two distinct points. The tangent-secant variety of X, $S_3^{ts}(X)$, is the closure of the union of all tangent-secant lines of X.

In proposition 4.2 it is shown that under the hypothesis of theorem 5.1, $S_3^{ts}(X)$ is set theoretically the quadratic envelope $QE(X)$ of X, i.e. the subvariety defined by the quadratic polynomials vanishing on X. In [11], Kwak shows that $QE(X) = S_3(X)$, where $S_3(X)$ is the trisecant variety. Our contribution is to show that $S_3^{ts}(X) = S_3(X)$. In proposition 4.1 it is shown that $S_3^t(X) = S_3(X)$, where $S_3^t(X)$ is the union of all tangent trisecant lines of X. The next step is to use the second fundamental form to control the dimension of the subvariety $S_3^{ns}(X) \subset S_3^t(X)$, which is the closure of the union of all trisecant lines of X that meet X only at one point. The end result is that non-constant tangentially homogeneous polynomials on X must vanish on the quadratic envelope of X. This and basic results on tangentially homogeneous polynomials finish the cases of quadratic index 0 and 1.

Notations:

Let $p_{[\,]} : \mathbb{C}^{N+1} \setminus \{(0, ..., 0)\} \to \mathbb{P}^N$ be the natural projection. If $V \subset \mathbb{P}^N$ is a subset, then $\hat{V}$ denotes the cone in $\mathbb{C}^{N+1}$ lying over V, i.e. $\hat{V} = \overline{p_{[\,]}^{-1}(V)}$.

Let $X \subset \mathbb{P}^N$ be a smooth projective subvariety and $x \in X$, $T_x X$ denotes the affine tangent space to X at x and $\mathbb{T}_x X \subset \mathbb{P}^N$ denotes the embedded projective tangent space to X at x.

Some of our results stem from local properties which hold at the general point x of the subvariety $X \subset \mathbb{P}^N$ being considered. These local properties are better expressed using particular local affine coordinates.

Let $x \in \mathbb{P}^N$, $X \subset \mathbb{P}^N$ a subvariety and $\{P_1, ..., P_k\}$ a finite collection of homogeneous polynomials in $\mathbb{C}[X_0, ..., X_N]$. We call a system of homogeneous coordinates $(X_0, ..., X_N)$ of $\mathbb{P}^N$:

(1) *adapted to* x, if $x = [1 : 0 : ... : 0]$.
(2) *adapted to* (X, x), if $x = [1 : 0 : ... : 0]$ and $\mathbb{T}_x X = V(X_{k+1}, X_{k+2}, ...X_N)$ for some k.
(3) *adapted to* $(X, x, \{P_i\})$, if it is adapted to (X, x) and X_0 does not divide any of the $P_i(X_0, ..., X_k, 0, ..., 0)$.

If $(X_0, ..., X_N)$ is adapted to x, then the affine neighborhood $U_0 = \{X_0 \neq 0\}$ of x is identified with $\mathbb{C}^N$ and has affine coordinates $x_i = X_i/X_0$, $i = 1, ..., N$ ($x = (0, ..., 0)$). The affine tangent space $T_x X$ will be often viewed as a linear subspace of U_0 and if $(X_0, ..., X_N)$ is adapted to (X, x), then $T_x X = V(x_{k+1}, ..., x_N)$. Note that $\mathbb{T}_x X \cap U_0 = T_x X$.

Let H be a homogeneous polynomial of degree d, $H \in \mathbb{C}[X_0, ..., X_N]^{(d)}$. We call $h = H/X_0^d$ a *dehomogenization of H adapted* to (X, x), if X_0 is the first coordinate of a system adapted to (X, x, H). Note $h \in \mathbb{C}[x_1, ..., x_N]$ with deg(h) =deg$H$.

A *dehomogenization of H* adapted $T_x X$ is the restriction:

$$h_x = h|_{T_x X} \tag{1.1}$$

where h is a dehomogenization of H adapted (X, x, H). Note that $h_x \in \mathbb{C}[x_1, ..., x_k]$ and if non-trivial, then deg$h_x = d$ with k as in 2). Also let:

$$h_x = h_x^{(0)} + ... + h_x^{(d)} \tag{1.2}$$

be the expansion of h_x by degrees, with $h_x^{(j)}$ the homogeneous part of degree j.

2. Symmetric twisted differentials and tangentially homogeneous polynomials

Bogomolov and the first author in [3] gave a geometric characterization of symmetric twisted differentials. In this section we give an algebraic characterization of symmetric twisted differentials, introducing the notion of tangentially homogeneous polynomials. This new approach is well suited to be used in conjunction with the results on projective differential geometry by Griffiths and Harris [8] concerning the relation between the properties of the tangent map and the second fundamental form, see next section. The main consequence of this interaction is the confirmation of conjecture 1.1 in the case of complete intersections.

Definition 2.1. *Let $L \subset \mathbb{P}^N$ be a linear subspace of dimension n and $x \in L$. A homogeneous polynomial $H \in \mathbb{C}[X_0, ..., X_N]$ is said to be homogeneous on L relative to x if there are homogeneous coordinates X'_0,...,X'_N on $\mathbb{P}^N$, such that:*

1. $L = V(X'_{n+1}, ..., X'_N)$
2. $x = [1 : 0 : ... : 0]$
3. $H|_L \in \mathbb{C}[X'_1, ..., X'_n]$

Geometrically H being homogeneous on L relative to x says that $V(H|_L)$ as a subvariety of L is the cone with vertex x over $Y \subset V(X'_0, X'_{n+1}, ..., X'_{n+1})$, with $Y = V(H|_L)$ where here $H|_L$ is viewed as polynomial in X'_1,...,X'_n as opposed to a polynomial in X'_0,...,X'_n.

Definition 2.2. *A homogeneous polynomial of degree d, $H \in \mathbb{C}[X_0, ..., X_N]^{(d)}$, is said to be tangentially homogeneous relative to $X \subset \mathbb{P}^N$ if one of the two following equivalent statements holds:*

1. *$\forall x \in X$, H is homogeneous on $\mathbb{T}_x X$ relative to x*
2. *$\forall x \in X$ the dehomogenizations of H adapted to $T_x X$, h_x, if non-trivial are homogeneous of degree d, i.e. $h_x = h_x^{(d)}$ in the notation of (1.2).*

The vector space of all tangentially homogeneous polynomials relative to X of degree d is denoted by $\mathbb{C}[X_0, ..., X_N]^{(d)}_{TX}$.

It is immediate from the definition that the product of two tangentially homogeneous polynomials relative to X is again tangentially homogeneous relative to X.

Definition 2.3. *The graded subalgebra of $\mathbb{C}[X_0, ..., X_N]$ whose graded pieces consist for each degree $m \in \mathbb{N}_0$ of $\mathbb{C}[X_0, ..., X_N]^{(m)}_{TX}$ is denoted by*

$$\mathbb{C}[X_0, ..., X_N]^{h}_{TX} = \bigoplus_{m \in \mathbb{N}_0} \mathbb{C}[X_0, ..., X_N]^{(m)}_{TX}$$

and is called the algebra generated by the tangentially homogeneous polynomials relative to X.

Example 2.1. *The constant polynomials are clearly tangentially homogeneous relative to any $X \subset \mathbb{P}^N$.*

Remark 2.1. *The tangentially homogeneous polynomials relative to X of positive degree must vanish on X.*

Example 2.2. *Any homogeneous polynomial F in the ideal of the tangent variety of X, $F \in I(Tan(X))$ is trivially tangentially homogeneous relative to X.*

Example 2.3. *A linear polynomial L is tangentially homogeneous relative to X if and only if $X \subset V(L)$.*

Example 2.4. *The key example of tangentially homogeneous polynomials relative to X are the quadratic polynomials $Q \in I(X)$. This holds, since $\forall x \in X$ $q_x^{(0)} = 0$ (Q vanishes on X) and $q_x^{(1)} = 0$ ($T_x X \subset T_x V(Q)$) making $q_x = q_x^{(2)}$ homogeneous of degree two.*

Proposition 2.1. *Let $X \subset P^N$ be a smooth subvariety and $F \in \mathbb{C}[X_0, ..., X_N]^{(m)}$. Then $F \in \mathbb{C}[X_0, ..., X_N]^{(m)}_{TX}$ if and only if for the general $x \in X$ F is homogeneous on $\mathbb{T}_x X$ relative to x.*

Proof. The result follows since the locus

$$L^h_{TX}(F) := \{x \in X |\ F \text{ is homogeneous on } \mathbb{T}_x X \text{ relative to } x\}$$

is an analytic subvariety of X and hence closed.

□

The following proposition will be useful in a later sections.

Proposition 2.2. *Let $X \subset \mathbb{P}^N$ be a smooth subvariety whose tangent variety Tan(X)=$\mathbb{P}^N$. If $F, G \in \mathbb{C}[X_0, ..., X_N]$ are nontrivial, homogeneous and $FG \in \mathbb{C}[X_0, ..., X_N]^h_{TX}$, then both F and G are in $\mathbb{C}[X_0, ..., X_N]^h_{TX}$.*

Proof. Suppose F is not tangentially homogeneous relative to X, then by proposition 2.1 f_x is not homogeneous at the general point $x \in X$. The condition that $Tan(X) = \mathbb{P}^N$ implies that G does not vanish on $\mathbb{T}_x X$. The result follows from $(fg)_x = f_x g_x$ and the product of two nontrivial polynomials being homogeneous implies that both factors are homogeneous. □

Remark 2.2. *If X is such that the tangent map τ is not surjective, then the algebra $\mathbb{C}[X_0, ..., X_N]^h_{TX}$ is not finitely generated. The simplest example to illustrate this is to consider a line $l \in \mathbb{P}^2$, say $l = \{X_0 = 0\}$. The algebra $\mathbb{C}[X_0, X_1, X_2]^h_{Tl} = \mathbb{C} + (X_0)$ which is not a finitely generated algebra. However, the conjecture 1.1 along with Theorem 3.1 for complete intersections state that this algebra is finitely generated when $n > 2/3(N-1)$ and X is nondegenerate. We will also show that the finite generation of $\mathbb{C}[X_0, ..., X_N]^h_{TX}$ holds for arbitrary non-degenerate X of codimension two if $\dim X \geq 3$.*

Theorem 2.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a nondegenerate smooth subvariety with $n > max\{1, 2/3(N-1)\}$. Then there is a graded isomorphism of algebras induced by the tangent map:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[X_0, ..., X_N]^h_{TX}$$

Proof. Recall that for a smooth projective subvariety $X \subset \mathbb{P}^N$ of dimension n we have the Gauss map $\gamma_X : X \to G(n+1, N+1)$. We define:

$$\widetilde{\Omega}^1_X = \gamma_X^* \mathcal{S}^\vee \otimes \mathcal{O}_X(-1) \tag{2.1}$$

where $\mathcal{S}^\vee$ is the dual of the universal subbundle on $G(n+1, N+1)$. We will consider $\widetilde{\Omega}^1_X(1)$, the *extended cotangent bundle over* X, which fits in the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N^*_{X/\mathbb{P}^N}(1) \arrow[r, "id"] \arrow[d] & N^*_{X/\mathbb{P}^N}(1) \arrow[d] & & \\
0 \arrow[r] & \Omega^1_{\mathbb{P}^N}|_X(1) \arrow[r] \arrow[d] & \bigoplus_{i=1}^{N+1} \mathcal{O}_X \arrow[r] \arrow[d, "q"'] & \mathcal{O}_X(1) \arrow[r] \arrow[d, "id"'] & 0 \\
0 \arrow[r] & \Omega^1_X(1) \arrow[r] \arrow[d] & \widetilde{\Omega}^1_X(1) \arrow[r] \arrow[d] & \mathcal{O}_X(1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The surjection $q : \bigoplus_{i=1}^{N+1} \mathcal{O}_X \to \widetilde{\Omega}^1_X(1) \to 0$ induces an inclusion of the corresponding projective bundles of hyperplanes:

$$f_q : \mathbb{P}(\widetilde{\Omega}^1_X(1)) \hookrightarrow X \times \mathbb{P}^N$$

A key geometric observation is that the inclusion f_q identifies the projective bundle $\mathbb{P}(\widetilde{\Omega}^1_X(1))$ with a projective subbundle of $X \times \mathbb{P}^N$ whose fiber over $x \in X$ is sent via the second projection to the embedded tangent space $\mathbb{T}_x X$ to X at x. This projective subbundle is usually denoted by $\mathbb{T}X$ and the the second projection restricted to $\mathbb{T}X$ is called the tangent map of X, $\tau : \mathbb{T}X \to \mathbb{P}^N$, see [8]. Note that using the identification given by f_q one can see the tangent map also as:

$$\begin{tikzcd}
\mathbb{P}(\widetilde{\Omega}^1_X(1)) \arrow[r, "\tau"] \arrow[d, "p"'] & \mathbb{P}^N \\
X \arrow[ur, hook, "i"] &
\end{tikzcd}$$

The yoga of projective bundles gives that τ induces an isomorphism $\mathcal{O}_{\mathbb{P}(\widetilde{\Omega}^1_X(1))}(m) \cong \tau^* \mathcal{O}_{\mathbb{P}^N}(m)$. The dimensional hypothesis $n > \max\{1, 2/3(N-1)\}$ is there to guarantee that the tangent map is both surjective and connected, where connectedness is the more delicate condition, see [3] for the proof. These two properties of the tangent map τ imply that the pullback τ^* gives natural isomorphisms:

$$H^0(X, S^m(\widetilde{\Omega}^1_X(1))) \cong H^0(\mathbb{P}(\widetilde{\Omega}^1_X(1)), \mathcal{O}_{\mathbb{P}(\widetilde{\Omega}^1_X(1))}(m)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

The above isomorphisms state that all twisted symmetric extended differentials of degree m, i.e. sections of $S^m[\widetilde{\Omega}^1_X(1)]$ over X, come in a natural geometric way from homogeneous polynomials of degree m in $\mathbb{C}[X_0, ..., X_N]$.

The next step is to identify which homogeneous polynomials correspond to twisted symmetric differentials. We will see that these polynomials are exactly the tangentially homogeneous polynomials relative to X.

The exact sequence $0 \to \Omega^1_X(1) \to \widetilde{\Omega}^1_X(1) \to \mathcal{O}_X(1) \to 0$ gives two maps of projective bundles. One map is the inclusion:

$$s : \mathbb{P}(\mathcal{O}_X(1)) \to \mathbb{P}(\widetilde{\Omega}^1_X(1))$$

Using the natural identification of X with $\mathbb{P}(\mathcal{O}_X(1))$, s is the section of $p : \mathbb{P}(\widetilde{\Omega}^1_X(1)) \to X$ that satisfies $\tau(s(x)) = i(x)$. The other map is the dominant rational map:

$$\pi : \mathbb{P}(\widetilde{\Omega}^1_X(1)) \dashrightarrow \mathbb{P}(\Omega^1_X(1)) \tag{2.2}$$

which fiberwise, $\pi_x := \pi|_{p^{-1}(x)}$, is the projection with center $s(x) \in p^{-1}(x)$. The indeterminacy locus of π is $s(X)$. These two maps fit into the commutative diagram

$$\begin{tikzcd}
\mathbb{P}(\Omega^1_X(1)) \arrow[dr, "p'"'] & \mathbb{P}(\widetilde{\Omega}^1_X(1)) \arrow[l, dashed, "\pi"'] \arrow[r, "\tau"] \arrow[d, shift left, "p"] & \mathbb{P}^N \\
& X \arrow[u, bend left, "s"] \arrow[ur, hook, "i"'] &
\end{tikzcd}$$

The dominant map (2.2) induces an inclusion:

$$\pi^* : H^0(\mathbb{P}(\Omega^1_X(1)), \mathcal{O}_{\mathbb{P}(\Omega^1_X(1))}(m)) \hookrightarrow H^0(\mathbb{P}(\widetilde{\Omega}^1_X(1)), \mathcal{O}_{\mathbb{P}(\widetilde{\Omega}^1_X(1))}(m))$$

which is just the natural inclusion of the space twisted symmetric m-differentials in the space of twisted extended symmetric m-differentials. A twisted extended symmetric m-differential $\widetilde{w}$ on X corresponds to a twisted symmetric m-differential w on X if and only if $\widetilde{w}$ is in the image of π^*.

A twisted extended symmetric m-differential $\widetilde{w}$ is in the image of π^* if and only if its restriction to each fiber of $\mathbb{P}(\widetilde{\Omega}^1_X(1))$, $\widetilde{w}|_{\mathbb{T}_x X}$, is in the image of π_x^*. The identifications described above give π_x geometrically as the projection on $\mathbb{T}_x X$ with center x:

$$\pi_x : \mathbb{T}_x X \dashrightarrow \mathbb{P}_l(T_x X)$$

where $\mathbb{P}_l(T_x X) = \mathbb{P}(\Omega^1_{x,X})$ is the projective space of lines through x. The image of the natural map

$$\pi_x^* : H^0(\mathbb{P}_l(T_x X), \mathcal{O}(m)) \to H^0(\mathbb{T}_x X, \mathcal{O}(m))$$

consists of the homogeneous polynomials on $\mathbb{T}_xX$ which are also homogenous relative to x (as in definition 2.2). Hence the twisted extended symmetric m-differential $\tilde{w}_H \in H^0(X, S^m\widetilde{\Omega}^1_X(1))$ associated with a degree m homogeneous polynomial $H \in \mathbb{C}[X_0, ..., X_N]$ corresponds to a twisted symmetric m-differential $w_H \in H^0(X, S^m\Omega^1_X(1))$ if and only if $\forall x \in X$, H is homogeneous on $\mathbb{T}_xX$ relative to x. In other words, $H \in \mathbb{C}[X_0, ..., X_N]^{(m)}_{TX}$.

□

In the proof above, the condition on the dimension of X is only used to guarantee that the tangent map is both connected and surjective.

Corollary 2.1. *(from the proof) Let $X \subset \mathbb{P}^N$ be a smooth subvariety for which the tangent map is both surjective and connected. Then*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega_X(1)]) \simeq \mathbb{C}[X_0, ..., X_N]^h_{TX}$$

3. Complete intersections

The goal of this section is to prove the conjecture 1.1 when X is a complete intersection. It will be shown that for a complete intersection X with the tangent variety $Tan(X) = \mathbb{P}^N$ any tangentially homogeneous polynomial relative to X must be in the algebra generated by the quadratic polynomials vanishing on X. A key ingredient in the argument is the relationship that exists between the equality $Tan(X) = \mathbb{P}^N$ and the algebraic independence of the quadratic forms on T_xX coming from the projective second fundamental $II_{X,x}$ at a general point $x \in X$.

Theorem 3.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate smooth complete intersection with $n > \max\{2/3(N-1), 1\}$. Then:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq Sym(H^0(\mathbb{P}^N, \mathcal{I}_X(2)))$$

.

Proof. It was shown in theorem 2.1 that the algebra of symmetric twisted differentials on X and the algebra generated by tangentially homogeneous polynomials relative to X are isomorphic via:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \xleftarrow{\tau^*} \mathbb{C}[X_0, ..., X_N]^h_{TX}$$

We need to show that the algebra $\mathbb{C}[X_0, ..., X_N]^h_{TX}$ is generated by any basis of $H^0(\mathbb{P}^N, I_X(2))$.

Let $c = \mathrm{codim}(X)$ and X of multi-degree $(d_1, ..., d_c)$, $d_1 \geq d_2 \geq ... \geq d_c$ with $I(X) = (F_1, ..., F_c)$ with $\deg F_i = d_i$. Note that, if $k = \min\{i | d_i = 2\}$, then $\{F_k, ..., F_c\}$ form a basis for $H^0(\mathbb{P}^N, I_X(2))$.

Let $H \in \mathbb{C}[X_0, ..., X_N]^h_{TX}$ have degree d. The goal is to show that $H \in \mathbb{C}[F_k, ..., F_c]$, $k = \min\{i | d_i = 2\}$. As observed earlier the tangentially homogeneous polynomial relative to X, H, must vanish on X, i.e. $H \in I(X)$. This allows us the following representation of H in terms of the defining equations of X:

$$H = \sum_{(i_1,...,i_c)\in I} G_{i_1...i_c} F_1^{i_1} ... F_c^{i_c} \tag{3.1}$$

where $I \subset \mathbb{Z}^c_{\geq 0}$ is some finite index set, $G_{i_1...i_c} \notin I(X)$ and $\deg(G_{i_1...i_c}) = d - (i_1 d_1 + \cdots + i_c d_c)$.

Let $\{f_{1,x}, ..., f_{c,x}\}$ be the dehomogenezation of $\{F_1, ..., F_c\}$ adapted to $T_x X$ with respect to a fixed homogeneous system $(X_0, ..., X_N)$ adapted to $(X, x, F_1, ..., F_c)$. As it was described in Example 2.4., the dehomogenizations along $T_x X$ for the quadratic polynomials F_i satisfy:

$$f_{i,x} = f^{(2)}_{i,x}, \qquad i \in \{k, ..., c\} \tag{3.2}$$

The same arguments give for any the dehomogenizations along $T_x X$ of $F \in I(X)$:

$$f^{(0)}_x = 0 \qquad f^{(1)}_x = 0 \tag{3.3}$$

Define:

$$ld(H) := 2 \min\{i_1 + \cdots + i_c | G_{i_1...i_c} \not\equiv 0\}$$

This is the lowest possible degree that can appear in the Taylor expansion h_x for any x. Now, consider the term of h_x of this minimal degree $ld(H)$,

$$h^{(ld(H))}_x = \sum_{i_1+...+i_c = ld(H)/2} g^{(0)}_{i_1...i_c,x} (f^{(2)}_{1,x})^{i_1} ... (f^{(2)}_{c,x})^{i_c} \tag{3.4}$$

The next step consists of showing that this term must be non-zero if x is a general point in X.

In fact, we do more, we show that the collection $\{f^{(2)}_{1,x}, ..., f^{(2)}_{c,x}\} \subset S^2[(T_x X)^*] = H^0(\mathbb{P}_l(T_x X), \mathcal{O}(2))$ is algebraically independent if x is general. The lemma below achieving this goal is a consequence of the work of Griffiths and Harris ([8] chapter 5) relating the geometry encoded in the second fundamental form of X and in the tangent map $t : \mathbb{T}X \to \mathbb{P}^N$.

Lemma 3.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate complete intersection, $X = V(F_1, ..., F_c)$, with $n > 2/3(N-2)$. If $x \in X$ is a general point, then the collection :*

$$\{{f_{1,x}}^{(2)}, ..., f_{c,x}^{(2)}\} \subset S^2[(T_xX)^*] = H^0(\mathbb{P}_l(T_xX), \mathcal{O}(2))$$

is algebraically independent. The $f_{i,x}^{(2)}$ are the quadratic terms of dehomogenizations adapted to T_xX of the F_i.

Proof. The proof involves properties of the second fundamental form of X. To understand how the second fundamental form comes into play and how to use it, we need to recall its definition and some of its distinct interpretations, see also [8] and [**?**] section 7.4.3.

If $X^{(n)} \subset \mathbb{P}^N$ is a subvariety, consider the Gauss map $\gamma_X : X_{sm} \to G(n+1, N+1)$, X_{sm} is the smooth part of X and $\gamma_X(x) = \hat{\mathbb{T}}_xX \subset \mathbb{C}^{N+1}$ the affine cone over $\mathbb{T}_xX$. The differential:

$$(d\gamma_X)_*(x) : T_xX \to T_{\hat{\mathbb{T}}_xX}G(n+1, N+1) \tag{3.5}$$

can be considered to be one of the versions of the second fundamental form of X at x.

Using the identification $T_{\hat{\mathbb{T}}_xX}G(n+1, N+1) = Hom(\hat{\mathbb{T}}_xX, \mathbb{C}^{N+1}/\hat{\mathbb{T}}_xX)$ we can rewrite 3.5 as $(d\gamma_X)_*(x) : T_xX \to Hom(\hat{\mathbb{T}}_xX/\hat{x}, \mathbb{C}^{N+1}/\hat{\mathbb{T}}_xX)$. Moreover, the natural identifications $\hat{\mathbb{T}}_xX/\hat{x} = T_xX$ and $\mathbb{C}^{N+1}/\hat{\mathbb{T}}_xX = N_xX$, where N_xX is the normal space to X at x, allow us to view 3.5 as $d\gamma_X(x) : T_xX \to Hom(T_xX, N_xX)$ or equivalently

$$(d\gamma_X)_*(x) : T_xX \otimes T_xX \to N_xX \tag{3.6}$$

A basic property of the bilinear form $(d\gamma_X)_*(x)$ in 3.6 is that it is symmetric hence can be viewed as an element in $Hom(S^2(T_xX), N_xX)$ or finally after dualizing one obtains the second fundamental form in the form that we work with:

$$II_{X,x} : N_xX^* \to S^2[(T_xX)^*] \tag{3.7}$$

After performing the previous yoga of identifications, we need to make concrete what the second fundamental form gives. Again here there are several choices on how to express the concrete meaning of $II_{X,x}$.

Even though the second fundamental form is local, we will describe it using some globally defined objects since it suits our purposes directly. Pick $H \in I(X)$ and let h_x be a dehomogenization of H at x. The gradient of h at x, $\text{grad}_x(h_x) \in (T_x\mathbb{P}^N)^*$, satisfies $T_xX \subset \text{Ker}(\text{grad}_x(h_x)) \subset T_x\mathbb{P}^N$, hence $\text{grad}_x(h_x) \in N_xX^*$ is well defined. The symmetric bilinear form on T_xX associated to H via the second fundamental form is:

$$II_{X,x}(\mathrm{grad}_x(h_x)) = h_x^{(2)}$$

note that $h_x^{(2)}$ is the restriction of the Hessian of h_x at x to T_xX and that this equality is only meaningful up to a multiplicative constant (which depends on the chosen dehomogenization). A word of caution, if $H \notin I(X)$ is such that $T_xX \subset \mathrm{Ker}(\mathrm{grad}_x(h_x)))$, then $\mathrm{grad}_x(h_x)) \in N_xX^*$ but $II_{X,x}(\mathrm{grad}_x(h_x))) \neq h_x^{(2)}$ in general (equality holds if and only if $V(H)$ osculates to order ≥ 2 at x in X).

The lemma follows from the results in [8] section 5 (a) describing how the second fundamental form provides information on the fibers of the tangent map $\tau : \mathbb{T}X \to \mathbb{P}^N$. In our case, the non-degeneracy of X and dimensional condition $n > 2/3(N-2)$ forces the tangent map to be surjective, hence the dimension of the general fiber of τ is $n - c$.

Let $z \in \mathbb{T}X$ be a general point with $p(z) = x$ a general point of X and $t(z) = y \in \mathbb{T}_xX \subset \mathbb{P}^N$. Consider the differential map:

$$(d\tau)_*(z) : T_z(\mathbb{T}X) \to T_y\mathbb{P}^N$$

The key point is that $(dp)_*(z)$ induces an isomorphism ([8] (5.5)):

$$\ker(d\tau)_*(z) \simeq \ker[(d\gamma_X)_*(x)(v_{\overline{xy}}, -)]$$

where $v_{\overline{xy}} \in T_xX$ is a vector in the direction of the line $\overline{xy}$ and $(d\gamma_X)_*(x)$ viewed as in (3.6). The subspace $\ker[(d\gamma_X)_*(x)(v_{\overline{xy}}, -)] \subset T_xX$ corresponds to all $u \in T_xX$ such that if one moves in X along the direction u the tangent spaces to X "preserve" $v_{\overline{xy}}$ to first order.

The next step is to understand $\ker[(d\gamma_X)_*(x)(v_{\overline{xy}}, -)]$ via (3.7). The image of $II_{X,x}$ is spanned by the set $\{f_{1,x}^{(2)}, ..., f_{c,x}^{(2)}\} \subset H^0(\mathbb{P}_l(T_xX), \mathcal{O}(2))$. This linear system defines a rational map

$$ii_{x,X} : \mathbb{P}_l(T_xX) \dashrightarrow \mathbb{P}^{c-1}$$

which satisfies $\dim \ker[(d\gamma_X)_*(x)(v_{\overline{xy}}, -)] = \dim \ker d(ii_{x,X})_*([v_{\overline{xy}}])$ for $[v_{\overline{xy}}]$ not in indeterminancy locus of $ii_{x,X}$ ([8] (5.6)).

Finally, assume $\{f_{1,x}^{(2)}, ..., f_{c,x}^{(2)}\}$ were algebraically dependent, then $\dim ii_{X,x}(\mathbb{P}_l(T_xX)) < c-1$. Hence for general $z \in \mathbb{T}X$ (and x and y as before):

$$\dim \ker(d\tau)_*(z) = \dim \ker[(d\gamma_X)_*(x)(v_{\overline{xy}}, -)] = \dim \ker(ii_{X,x})_*([v_{\overline{xy}}]) > n - c$$

contradicting the surjectivity of the tangent map τ.

$\square$

Returning to the proof of the theorem, the above lemma gives that any $H \in I(X)$ is such that for a general point $x \in X$ the term of Taylor expansion of h_x of degree $ld(H)$, see (3.4), is non-vanishing.

The polynomial H being a nontrivial tangentially homogeneous relative to X and $\mathrm{Tan}(X) = \mathbb{P}^N$ gives that (the general dehomogenization relative to T_xX) h_x is homogeneous of degree d for general $x \in X$ and hence the following must hold:

$$d = ld(H)$$

The definition of $ld(H)$ tell us that the polynomials $G_{i_1 \ldots i_c}$ in (3.1) are non-vanishing only if $2(i_1 + \ldots + i_c) \geq d$. Combining with $d \geq d_1 i_1 + \ldots + d_c i_c$, $d_1, \ldots, d_{k-1} > 2$ and $d_k, \ldots, d_c = 2$, one obtains that:

$$G_{i_1 \ldots i_c} \neq 0 \qquad \text{only if } i_1 = \ldots = i_{k-1} = 0$$

and

$$\deg G_{i_1 \ldots i_c} = 0$$

In other words,

$$H = \sum_{i_k + \ldots + i_c = d/2} c_{i_k \ldots i_c} F_k^{i_k} \ldots F_c^{i_c} \in \mathbb{C}[F_k, \ldots, F_c]$$

as desired. □

The proof of theorem 3.1 also gives,

Corollary 3.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a smooth complete intersection with $Tan(X) = \mathbb{P}^N$. Then:*

$$\mathbb{C}[X_0, \ldots, X_N]^h_{TX} = \mathbb{C}[Q_0, \ldots, Q_r]$$

where $\{Q_0, \ldots, Q_r\}$ is any basis of $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.

In section 5 we provide an example where the conjecture holds despite X not being a complete intersection (it will be in the range $2/3(N-1) < \dim X \leq 2/3N$).

4. Tangent-secant varieties, trisecant varieties and the quadratic envelope

There are three auxiliary varieties associated to a subvariety $X \subset \mathbb{P}^N$: the trisecant variety $S_3(X)$, the tangent-secant variety $S_3^{ts}(X)$ and the quadratic envelope $QE(X)$, that play a role in the study of tangential homogeneous polynomials. This section describes some of the interplay between these three varieties with special emphasis on the codimension two case. We show that $S_3^{ts}(X) = QE(X)$ which provide us with a path to prove the conjecture 1.1 in codimension two, by allowing us to deal with the cases that are not known to be complete intersections.

A tangentially homogeneous polynomial H relative to X must vanish on any line that is trisecant of the type tangent-secant, i.e. a line that is tangent and meets X at least in two distinct points. The variety which is the closure of the union of all tangent-secant lines is called the tangent-secant variety of X, $S_3^{ts}(X)$. Hence any tangentially homogeneous polynomial $H \in \mathbb{C}[X_0, ..., X_N]^h_{TX}$ must satisfy:

$$S_3^{ts}(X) \subset V(H)$$

The tangent-secant variety lies inside the trisecant variety, $S_3^{ts}(X) \subset S_3(X)$. Bezout's theorem implies another natural inclusion, that of the trisecant variety X inside the quadratic envelope of X, $QE(X)$ (the variety defined by the quadratic polynomials vanishing on X),

$$S_3(X) \subset QE(X)$$

First we show that the trisecant-tangent variety, $S_3^t(X)$ (union of all trisecant lines to X that are also tangent) coincides with the trisecant variety when the codimension is low.

Proposition 4.1. *Let $X^{(n)} \subset \mathbb{P}^N$ be a smooth subvariety with $n > \max\{2/3(N-1), 1\}$. Then*

$$S_3^t(X) = S_3(X)$$

Proof. The inclusion $X \subset S_3^t(X)$ follows from $\dim(\mathbb{T}_x X \cap X) \geq 1$ since then any line joining x to a distinct $y \in \mathbb{T}_x X \cap X$ is a trisecant line which is tangent. The inequality $\dim(\mathbb{T}_x X \cap X) \geq 1$ holds if $n \geq 1/2(N+1)$ which is satisfied if $n > \max\{2/3(N-1), 1\}$. It remains to show that $z \in S_3(X) \setminus X$ must be also in $S_3^t(X)$.

Let $z \in S_3(X) \setminus X$ and l a trisecant line to X passing through z. If l is tangent to X, then $z \in S_3^t(X)$; otherwise l meets X at 3 distinct points. Consider the subvariety of trisecant lines to X passing through z,

$$\Sigma_3(X, z) = \{l \in \mathbb{G}(1, N) | \text{length}(X \cap l) \geq 3\} \subset \mathbb{G}(1, N)$$

Claim: If there is a $l \in \Sigma_3(X, z)$ not tangent to X, then $\Sigma_3(X, z)$ has irreducible components that are positive dimensional.

The proposition follows from the claim since lemma 1.9 of [3] guarantees that one the trisecant lines in $\Sigma_3(X, z)$ must be tangent. The mentioned lemma states that if a projective family of lines in $\mathbb{P}^N$ is such that all its lines meet X at least twice and pass through a fixed point not in X (here z) and the family does not consists of a single line, then there must be a tangent line in the family.

(proof of claim): Let $H \subset \mathbb{P}^N \setminus \{z\}$ be an hyperplane. Consider the projection

$$p_z : \mathbb{P}^N \dashrightarrow H$$

with center z into H. The projection $p_z|_X$ is a finite map with $p_z(X) \subset H$ irreducible of dimension n.

Let $y = l \cap H \in p_z(X)$ and x_1, x_2, x_3 be distinct points in $l \cap X$. There are analytical neighborhoods $U_y \subset H$ and $U_i \subset X$ of respectively y and the x_i, $i = 1, 2, 3$ such that $Z_i := p_z(U_i)$ are irreducible n-dimensional subvarieties of U_y, [**?**] section E-7. Using the intersection dimension inequality, [**?**] section G-10, we obtain that if W is a germ based at y of any irreducible component of $Z_1 \cap Z_2 \cap Z_3$, then:

$$\dim W \geq 3n - 2(N-1) \tag{4.1}$$

The lines through z and $y' \in Z_1 \cap Z_2 \cap Z_3 \subset H$ intersect X at least 3 times. Note that $\Sigma_3(X, z)$ can be viewed as a subvariety of H, since the variety of lines in $\mathbb{P}^N$ through z is canonically identified with H. This identification allows us to see $Z_1 \cap Z_2 \cap Z_3 \subset \Sigma_3(X, z)$. It follows from the dimensional condition $n > 2/3(N-1)$ and 4.1 that at least one irreducible component of $\Sigma_3(X, z)$ has dimension ≥ 1 and the claim follows

$\square$

The rest of this section pertains to the codimension two case, the goal is show that if $X^{(n)} \subset \mathbb{P}^{n+2}$ is smooth and $n \geq 3$, then

$$S_3^{ts}(X) = S_3(X) = QE(X)$$

This allows us in the next section to prove the conjecture 1.1. for codimension two.

Due to the work of Severi, Segre, Ran and Kwak, see [13] and [11], it is known that in codimension two, $S_3(X) = QE(X)$ if $\dim X \geq 3$. This is a consequence of the classification of varieties with many lines plus the work of Ran and Kwak where it is shown that if $X^{(n)} \subset \mathbb{P}^{n+2}$ for $n \geq 3$ is not contained in a quadric, then its trisecant variety is the full $\mathbb{P}^{n+2}$ (Ran showed the case $n > 3$ and Kwak the case $n = 3$), see 3.6 of [11]. Note that $S_3(X) = QE(X)$ in codimension two implies in particular that $S_3(X)$ is irreducible which is important in what follows.

Proposition 4.1 then gives that

$$S_3^t(X) = QE(X)$$

For our purposes, we need to show that $S_3^{ts}(X) = QE(X)$. To achieve this we consider another auxiliary variety, $S_3^{ns}(X)$, the "non-secant" trisecant variety of X. $S_3^{ns}(X)$ is the closure of the union of all "non-secant" trisecant lines to X (lines that meet X only at a single point), which is interesting in its own right.

Let $|II_{X,x}|$ be the linear system of quadrics on $\mathbb{P}_l(T_xX) = \mathbb{P}(\Omega^1_{X,x})$ associated with the second fundamental form. A subvariety $X^{(n)} \subset \mathbb{P}^{n+2}$ has a stratification $X = X_g \cup X_s \cup X_{vs}$ where:

$$X_g := \{x \in X | \dim |II_{X,x}| = 1\}$$

$$X_s := \{x \in X | \dim |II_{X,x}| = 0\}$$

$$X_{vs} := \{x \in X || II_{X,x}| = \emptyset\}$$

The points in X_g, X_s and in X_{vs} will be called somewhat general points, special points and very special points, respectively. The strata X_{vs} is a subvariety of X and so is $X_s \cup X_{vs}$.

The next lemma describes, for $x \in X$ general, the tangent cone $C_x(T_xX \cap X)$, when X is a smooth threefold in $\mathbb{P}^5$ that is not a $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ Segre embedded, i.e a Segre cubic scroll. The information on the tangent cone $C_x(T_xX \cap X)$ is used to control the size of the "non-secant" trisecant variety of X, a necessary result to show that the tangent secant variety of X is the quadratic envelope.

Lemma 4.1. *Let $X^{(3)} \subset \mathbb{P}^5$ be a smooth subvariety and $x \in X$ a general point. Then one of the following holds:*

i) The tangent cone $C_x(T_xX \cap X)$ is the union of 4 lines.

ii) X is a Segre cubic scroll.

Proof. At the general point $x \in X$ the second fundamental form gives a pencil of conics, $|II_{X,x}|$, in $\mathbb{P}_l(T_xX) = \mathbb{P}^2$. As seen before, these conics are described by the degree 2 terms of the dehomogenizations adapted to T_xX of the elements in $I(X)$. Let $f^{(2)}_{1,x}$ and $f^{(2)}_{2,x}$ be the degree 2 terms of the dehomogenizations adapted to T_xX of 2 polynomials F_1 and F_2 locally defining X at x.

The tangent cone $C_x(T_xX \cap X) \subset T_xX$ is defined (as a subvariety of T_xX) by the ideal $I^{in}_x(T_xX \cap X)$ generated by initial terms of the Taylor expansions at x of the defining equations of $T_xX \cap X$.

The dichotomy in the lemma sprouts from the existence or nonexistence of a common linear factor in the quadratic terms $f^{(2)}_{1,x}$ and $f^{(2)}_{2,x}$ described above. If there is no common linear factor, then:

$$I^{in}_x(T_xX \cap X) = < f^{(2)}_{1,x}, f^{(2)}_{2,x} >$$

and hence the base locus of $|II_{X,x}|$ consists of 4 points and $C_x(T_xX \cap X)$ is the union of 4 lines.

If there is a common linear factor in the quadratic terms $f^{(2)}_{1,x}$ and $f^{(2)}_{2,x}$ (which corresponds to a line as a fixed component of $|II_{X,x}|$), then all conics in $|II_{X,x}|$ are singular. This implies (see for example [7] 2.5.3) that the dual defect $\delta_*(X) = 1$ and hence the dual variety X^* of X is of the same dimension as X. The smooth subvarieties $X^{(n)} \subset \mathbb{P}^N$ with $\dim X = \dim X^*$ and $N \geq 3/2n$ are classified (see for example [14] 4.4.9). If X is a threefold in $\mathbb{P}^5$ with $\dim X = \dim X^*$, then X is the Segre cubic scroll.

$\square$

Proposition 4.2. *Let $X^{(n)} \subset \mathbb{P}^{n+2}$ be a smooth non-degenerate subvariety with dimension $n \geq 3$. Then*

$$S_3^{ts}(X) = QE(X)$$

Proof. We prove the case $n = 3$ and then show how to derive the result from this case.

If $X^{(3)} \subset \mathbb{P}^5$ is non-degenerate and smooth, then $S_3^{ts}(X) = QE(X)$.

Proof. (of claim) The proof will be partitioned according to value of the quadratic index of X in $\mathbb{P}^5$, $i_q(X) := h^0(\mathbb{P}^5, \mathcal{I}_X(2))$.

Case: $i_q(X) \geq 2$.

In this case the quadratic envelope $QE(X) = X$, since $i_q(X) \geq 2$ implies that X must be a complete intersection of quadrics or the Segre variety (also an intersection of quadrics), see for example 3.6 of [11] . The equality $S_3^{ts}(X) = X$ follows from $X \cap \mathbb{T}_x X$ being positive dimensional, which implies that through every $x \in X$ passes a tangent secant line.

Case: $i_q(X) = 1$.

Here $QE(X) = Q$, Q the unique irreducible quadric containing X (if Q was reducible, then X would be degenerate). The result follows by dimension considerations. It is enough to show that some component (and hence unique) of $S_3^{ts}(X)$ has dimension greater than $\dim X$. To obtain this it is enough to show that $\mathbb{T}_x X \cap X$ is not a cone with vertex x for general $x \in X$.

According to lemma 4.1 either the the projective tangent cone to $\mathbb{T}_x X \cap X$ at x, $\mathbb{C}_x(\mathbb{T}_x X \cap X)$, is the union of 4 lines or X is a Segre cubic scroll. The last case is not possible since $i_q(X) = 1$ and not 3.

Suppose at $x \in X$ general $\mathbb{T}_x X \cap X$ is a cone with vertex at x, then since $\mathbb{C}_x(\mathbb{T}_x X \cap X)$ is the union of 4 lines it follows $\mathbb{T}_x X \cap X$ is the same union of 4 lines. This implies X has degree 4 and hence $i_q(X) \neq 1$, a contradiction.

Case: $i_q(X) = 0$.

Need to show that $S_3^{ts}(X) = \mathbb{P}^5$ or equivalently that $S_3^{ns}(X) \neq \mathbb{P}^5$. A tangent trisecant line l that is non-secant (in the sense that it does not meet X at two distinct points) must be such that $[l] \in \mathbb{P}(\Omega^1_{X,x})$ belongs to the base locus $B(|II_{X,x}|)$ (such are the lines that meet X at x with multiplicity greater than 2).

The contributions to $S_3^{ns}(X)$ by each point in $x \in X$ are necessarily contained in subvarieties whose dimension have bounds depending on which strata X_g, X_s or X_{vs} x lies in.

A somewhat general point $x \in X_g$ according to lemma 4.1 is such that through x there are at most only 4 lines that are non-secant trisecant lines (since X can not be the Segre cubic scroll).

A special point $x \in X_s$ is such that the nonsecant trisecant lines passing through x must be contained in a quadric with vextex x in $\mathbb{T}_x X$, the quadric whose base is the conic in $\mathbb{P}_l(T_x X)$ determined by $|II_{X,x}|$.

A very special point $x \in X_{vs}$ is characterized by the property that all lines through x can be non-secant trisecant, hence the closure of the contribution to $S_3^{ns}(X)$ by x could be the full $\mathbb{T}_x X$.

The desired conclusion, $S_3^{ns}(X) \neq \mathbb{P}^5$, follows from the bounds on dimension of each strata. The strata X_g is a quasi-projective variety of dimension three. The contribution to $S_3^{ns}(X)$ by each point is at most 1-dimensional, hence the closure of the contribution from X_g to $S_3^{ns}(X)$ is at most 4-dimensional.

The quasi-projective variety X_s is at most two dimensional. Each $x \in X_s$ contributes at most with a 2-dimensional subvariety of $S_3^{ns}(X)$, making the contribution to $S_3^{ns}(X)$ coming from X_s at most 4-dimensional.

The subvariety X_{vs} is a finite collection of points. This follows from Zak's result on tangencies, see [16], which implies that the fibers of the Gauss map are 0-dimensional. Using the fact that the 2nd fundamental form is the differential of the Gauss map, it follows that the connected components of X_{vs} must be contained in the fibers of the Gauss map. As a consequence, the contribution to $S_3^{ns}(X)$ coming from X_{vs} is at most a finite union of 3-planes in $\mathbb{P}^5$.

Combining the contributions to $S_3^{ns}(X)$ coming from the three strata, one has that $\dim S_3^{ns}(X) < 5$ and the claim is settled. □

To terminate the proof of the proposition consider $L \subset \mathbb{P}^{(n+2)}$ a general 5-plane. The claim gives that $S_3^{ts}(X \cap L) = QE(X \cap L)$. Since $QE(X \cap L) = QE(X) \cap L$ holds, this implies that $S_3^{ts}(X)$ contains $QE(X) \cap L$ for the general 5-plane and hence $S_3^{ts}(X) = QE(X)$.

□

5. Codimension two

Our approach to conjecture 1.1 for $X \subset \mathbb{P}^N$ of codimension two is to stratify the problem according to the quadratic index $i_q(X) = h^0(\mathbb{P}^N, \mathcal{I}_X(2))$. The cases where $i_q(X) < 2$ where little is known about Hartshorne's conjecture can be handled due to:

Theorem 5.1. *Let $X^{(n)} \subset \mathbb{P}^{n+2}$ be a non-degenerate smooth subvariety. Then the tangent map induces the graded isomorphism:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[Q_0, ..., Q_r]$$

where $\{Q_0, ..., Q_r\}$ is any basis of $H^0(\mathbb{P}^N, I_X(2))$.

Proof. The proof is done by considering the three possibilities for the quadratic index $i_q(X)$.

Case $i_q(X) = 0$

The results mentioned earlier by Ran [13] and Kwak [11] give $S_3(X) = \mathbb{P}^{n+2}$ and hence by proposition 4.2 $S_3^{ts}(X) = \mathbb{P}^{n+2}$. This implies that the tangentially homogeneous polynomials relative to X of positive degree must be trivial. Therefore

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[X_0, ..., X_N]^h_{TX} = \mathbb{C}$$

as desired.

Case $i_q(X) = 1$

In this case $S_3(X) = V(Q)$ with Q irreducible (X non-degenerate) spanning $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$ (see [11]). Same argument gives $S_3^{ts}(X) = V(Q)$ which implies that Q divides all tangentially homogeneous polynomials relative to X, H, of positive degree. This is not enough, what is needed is that $H \in \mathbb{C}[Q]$. If H is irreducible, then $H = cQ$,with $c \in \mathbb{C}^*$. If H is reducible, then use proposition 2.2 which holds since the conditions: $\mathrm{cod}(X) = 2$ and $n \geq 3$ give that the tangent variety $\mathrm{Tan}(X) = \mathbb{P}^{n+2}$. Hence irreducible factors of H must also be tangentially homogeneous polynomials relative to X and we obtain the graded isomorphism:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[X_0, ..., X_N]^h_{TX} = \mathbb{C}[Q]$$

as desired.

Case $i_q(X) \geq 2$

The condition $i_q(X) \geq 2$ forces degree of X to be at most four. In this case the Hartshorne conjecture is established and the result follows from the theorem 3.1, except for $\Sigma_{1,2} \subset \mathbb{P}^5$, the Segre cubic scroll, which is not a complete intersection and is dealt with in an explicit fashion below.

i) $\dim X \geq 4$

Ziv Ran's result on the Harthshorne's conjecture for codimension 2 [13], states that if degree X$\leq \dim X$, then X is a complete intersection. Hence $X = V(Q_0, Q_1)$ with $\{Q_0, Q_1\}$ a basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$ and theorem 3.1 implies the graded isomorphism:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq \mathbb{C}[X_0, ..., X_N]^h_{TX} = \mathbb{C}[Q_0, Q_1]$$

as desired.

ii) $\dim X = 3$

In this case either X is a complete intersection or X is projectively equivalent to the Segre cubic scroll, $\Sigma_{1,2}$ (see 3.6 of [11]). In the first case, the result holds as above. The remaining case of $X = \Sigma_{1,2}$ is dealt with following a distinct approach.

Let $\Sigma_{1,2}$ be the Segre cubic scroll, i.e. the image of the Segre embedding $\sigma : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The scroll $\Sigma_{1,2}$ is not a complete intersection, the ideal $I(\Sigma_{1,2})$ is generated by 3 quadratic polynomials $\{Q_0, Q_1, Q_2\}$ forming a basis for $H^0(\mathbb{P}^2, \mathcal{I}_{\Sigma_{1,2}}(2))$. The approach we follow is not to directly show that $\mathbb{C}[X_0, ..., X_5]^h_{T\Sigma_{1,2}} = \mathbb{C}[Q_0, Q_1, Q_2]$ and then apply theorem 2.1. Instead we directly calculate the dimensions $h^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)])$ and check that

$$h^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) = \dim \mathbb{C}[Q_0, Q_1, Q_2]^{(m)}$$

which guarantees the desired isomorphism of algebras:

$$\bigoplus_{m=0}^{\infty} H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) \simeq \mathbb{C}[Q_0, Q_1, Q_2]$$

Let $p_i : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^i$ be the natural projections. Using the biholomorphism $\sigma : \mathbb{P}^1 \times \mathbb{P}^2 \to \Sigma_{1,2}$ and $\sigma^*\mathcal{O}_{\Sigma_{1,2}}(1) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)$, we have that:

$$\sigma^*(\Omega^1_{\Sigma_{1,2}}(1)) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1,1) \oplus p_2^*(\Omega^1_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)$$

and hence:

$$(5.1)\quad H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) \simeq \bigoplus_{i=0}^{m} H^0(\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega^1_{\mathbb{P}^2}]))$$

The summands of the right side of (5.1) do vanish if:

i) $i < m/2$, since on the fibers of p_2, $p_2^{-1}(t) = \mathbb{P}^1$, the bundle:

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega^1_{\mathbb{P}^2}])|_{\mathbb{P}^1} \simeq \mathcal{O}(-m+2i) \oplus ... \oplus \mathcal{O}(-m+2i)$$

has no nontrivial sections on $\mathbb{P}^1$.

ii) $i > m/2$, since on the fibers of p_1, $p_1^{-1}(t) = \mathbb{P}^2$, the bundle:

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i,m)\otimes p_2^*(S^i[\Omega^1_{\mathbb{P}^2}])|_{\mathbb{P}^2}\simeq(S^i\Omega^1_{\mathbb{P}^2})(m)$$

The claimed vanishing follows from:

$$H^0(X,(S^i\Omega^1_{\mathbb{P}^n})(m))=0 \tag{5.2}$$

if $m<2i$ (ours is the case $n=2$).

To see this, a symmetric differential $w\in H^0(X,(S^i\Omega^1_{\mathbb{P}^n})(m))$ defines at $x\in\mathbb{P}^n$ where $w(x)\neq 0$ an hypersurface:

$$Z_w(x)\subset T_x\mathbb{P}^n$$

consisting of all tangent vectors in the zero locus of $w(x)$, where $w(x)$ is viewed as an homogeneous polynomial of degree i on $T_x\mathbb{P}^n$ (with values in $\mathcal{O}_{\mathbb{P}^n}(m)|_x\simeq\mathbb{C}$).

If there is a nontrivial differential w, then at a general point $x\in\mathbb{P}^n$ there is a line $l\subset\mathbb{P}^n$ through x ($i_l:\mathbb{P}^1\hookrightarrow X$ with $i_l(\mathbb{P}^1)=l$) such that $T_xl\not\subset Z_w(x)$. This implies

$$0\neq(di_l)^*w\in H^0(\mathbb{P}^1,(S^i\Omega^1_{\mathbb{P}^1})(m))$$

contradicting $H^0(\mathbb{P}^1,(S^i\Omega^1_{\mathbb{P}^1})(m))\simeq H^0(\mathbb{P}^1,\mathcal{O}(-2i+m))=0$ when $m<2i$.

At this point we can conclude that:

(1) $H^0(\Sigma_{1,2},S^m[\Omega^1_{\Sigma_{1,2}}(1)])=0$, $m=odd$

(2) $H^0(\Sigma_{1,2},S^m[\Omega^1_{\Sigma_{1,2}}(1)])\simeq H^0(\mathbb{P}^2,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)])$, $m=even$.

The above gives that $h^0(\Sigma_{1,2},S^m[\Omega^1_{\Sigma_{1,2}}(1)])=\dim\mathbb{C}[Q_0,Q_1,Q_2]^m(=0)$ if m is odd. Hence what remains is to show the same equality for m even, to this end, it is enough to show that $h^0(\mathbb{P}^2,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)])\leq\binom{\frac{m}{2}+2}{2}=\dim\mathbb{C}[Q_0,Q_1,Q_2]^m$ (the last equality holds since Q_0,Q_1 and Q_2 are algebraically independent). Note that $h^0(\Sigma_{1,2},S^m[\Omega^1_{\Sigma_{1,2}}(1)])\geq\dim\mathbb{C}[Q_0,Q_1,Q_2]^m$ follows from theorem 2.1.

The inequality $h^0(\mathbb{P}^2,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)])\leq\binom{\frac{m}{2}+2}{2}$ holds due to $\Omega^1_{\mathbb{P}^2}(2)|_{\mathbb{P}^1}\simeq\mathcal{O}(1)\oplus\mathcal{O}$, the restriction exact sequence:

$$0\to S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]\otimes\mathcal{O}(-1)\to S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]\to S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1}\to 0$$

and (5.2). To see this, observe that $S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1}=O_{\mathbb{P}^1}(\frac{m}{2})\oplus O_{\mathbb{P}^1}(\frac{m}{2}-1)\oplus...\oplus\mathcal{O}$ and hence

$$h^0(\mathbb{P}^1,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1})=\binom{\frac{m}{2}+2}{2}$$

$h^0(\mathbb{P}^2,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)])\leq\binom{\frac{m}{2}+2}{2}$ holds since by (5.2), $h^0(\mathbb{P}^2,S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)])\otimes\mathcal{O}(-1))=0$.
$\square$

References

[1] F. Bogomolov. Families of curves on a surface of general type. *Doklady Akad. Nauk SSSR*, 236(5):1041–1044, 1977.

[2] F. Bogomolov and B. D. Oliveira. Hyperbolicity of nodal hypersurfaces. *J. Reine Angew. Math.*, 596:89–101, 2006.

[3] F. Bogomolov and B. D. Oliveira. Symmetric tensors and the geometry of subvarieties of $\mathbb{P}^n$. *Geometric and Functional Analysis*, 18:637–656, 2008.

[4] D. Brobek. Symmetric differential forms on complete intersection varieties and applications. *Math. Ann.*, 366(1-2):417–466, 2016.

[5] Y. Brunebarbe, B. Klingler, and B. Totaro. Symmetric differentials and the fundamental group. *Duke Math. J.*, 162(14):2797–2813, 2013.

[6] O. Debarre. Varieties with ample cotangent bundle. *Compositio Mathematica*, 141:1445–1459, 2005.

[7] G. Fischer and J. Piontkowski. *Ruled Varieties*. Vieweg, Berlin, 2001.

[8] P. Griffiths and J. Harris. Algebraic geometry and local differential geometry. *Annales Scientifiques de l'Ecole Normale Suprieure*, 12:355–452, 1979.

[9] R. Hartshorne. Varieties of small codimension in projective space. *Bulletin of the American Mathematical Society*, 80:1017–1032, 1974.

[10] P. Ionescu and F. Russo. Manifolds covered by lines and the hartshorne conjecture for quadratic manifolds. *American Journal of Mathematics*, 135:349–360, 2013.

[11] S. Kwak. Smooth threefolds in $\mathbb{P}^5$ without apparent triple or quadruple points and a quadruple-point formula. *Mathematische Annalen*, 320:649–664, 2001.

[12] J. M. Landsberg. On degenerate secant and tangential varieties and local differential geometry. *Duke Mathematical Journal*, 85:605–634, 1996.

[13] Z. Ran. On projective varieties of codimension 2. *Inventiones Mathematicae*, 73:333–336, 1983.

[14] F. Russo. *On the geometry of some special projective varieties*. Springer, Berlin, 2016.

[15] M. Schneider. Symmetric differential forms as embedding obstructions and vanishing theorems. *Journal of Algebraic Geometry*, 1:175–181, 1992.

[16] F. Zak. *Tangents and Secants of Algebraic Varieties*. American Mathematical Society, 1993.

Department of Mathematics, University of Miami, Coral Gables, FL 33146
E-mail address: `b.deolive@math.miami.edu`

Department of Mathematics, Pennsylvania State University, State College, PA 16803
E-mail address: `cyl5328@psu.edu`