

CONVEXITY OF COVERINGS OF PROJECTIVE VARIETIES AND VANISHING THEOREMS

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ABSTRACT. Let X be a projective manifold, $\rho : \tilde{X} \rightarrow X$ its universal covering and $\rho^* : Vect(X) \rightarrow Vect(\tilde{X})$ the pullback map for isomorphism classes of vector bundles. This article makes the connection between the properties of the pullback map ρ^* and the properties of the function theory on \tilde{X} . Our approach motivates a weakened version of the Shafarevich conjecture: the universal covering \tilde{X} of a projective manifold X is holomorphically convex modulo the pre-image $\rho^{-1}(Z)$ of a subvariety $Z \subset X$. We prove this conjecture for projective varieties X whose pullback map ρ^* identifies a nontrivial extension of a negative vector bundle V by \mathcal{O} with the trivial extension. We prove the following pivotal result: if a universal cover of a projective variety has no nonconstant holomorphic functions then the pullback map ρ^* is almost an imbedding. Our methods also give a new proof of $H^1(X, V) = 0$ for negative vector bundles V over a compact complex manifold X whose rank is smaller than the dimension of X .

1. INTRODUCTION

This paper deals with two questions about the function theory of universal covers \tilde{X} of projective varieties X . One question, appearing in section 2, is on the abundance of holomorphic functions on \tilde{X} . The main conjecture on the abundance is the Shafarevich uniformization conjecture (see below). The other question, appearing in section 3, is on the simple existence of nonconstant holomorphic functions on \tilde{X} . It is an open question to know whether the universal cover of a projective variety has nonconstant holomorphic functions. In dealing with both problems, we use the same idea. The idea is to explore the relation between the existence of nonconstant holomorphic functions on \tilde{X} and the identification on \tilde{X} of the pullback of distinct isomorphism classes of vector bundles on X . This relation gives a new approach to the production of holomorphic functions on universal cover \tilde{X} of a projective manifold X . Additionally, using the methods of section 2 we give a new proof of the vanishing of $H^1(X, V)$ for negative vector bundles V over

* Partially supported by NSF grant DMS-0100837. ** Partially supported by NSF Postdoctoral Research Fellowship DMS-9902393 and NSF grant DMS-0306487

a compact complex manifold X whose rank is smaller than the dimension of X . The following is a description of what can be found in this paper.

In section 2, we approach the abundance of holomorphic functions on the universal covers of projective varieties. In the early 70's I. Shafarevich proposed the following conjecture on the function theory of universal covers: The universal cover \tilde{X} of a projective variety X is holomorphically convex, i.e every discrete sequence of points of \tilde{X} has a holomorphic function on \tilde{X} that is unbounded on it. This conjecture has been proved in some cases (see [Ka95] and [EKPR] for the strongest results), but the general case has remained unreachable. For the general case, there is the work of Kollar [Ko93] on the existence of Shafarevich maps (Campana [Ca94] has dealt with Kahler case). The existence and properties of the Shafarevich maps do not give information on the existence of holomorphic functions on \tilde{X} . But they are an essential tool for dealing with and understanding the conjecture. There are two main reasons for the difficulty in proving the Shafarevich conjecture. The first reason is that the conjecture proposes that noncompact universal covers \tilde{X} have many holomorphic functions. But, on the other hand, there is a lack of methods to construct holomorphic functions on \tilde{X} . The second reason comes from the main geometric obstruction to holomorphic convexity. A holomorphic convex analytic space can not have an infinite chain of compact subvarieties. The existence of these infinite chains on universal covers of projective varieties has not been ruled out. In fact, the first author and L. Katzarkov produced some examples of algebraic surfaces that possibly contain infinite chains [BoKa98].

The possible existence of infinite chains on universal covers demands that one rethinks the Shafarevich conjecture. We do exactly that in section 2, where we use our approach for obtaining functions on the universal cover to motivate the following weakened Shafarevich conjecture.

Conjecture 2.3. *The universal covering \tilde{X} of a projective variety X is holomorphically convex modulo the pre-image of a subvariety $Z \subset X$.*

This means that for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z , there exists a holomorphic function f on \tilde{X} which is unbounded on the sequence. The interest of this conjecture is that it is still very strong but does not exclude the existence of infinite chains of compact subvarieties. The strength of our weakened conjecture is manifested in the fact that it would still separate universal covers of projective varieties from universal covers of compact non-kahler manifolds with many holomorphic functions. The example to have in mind is the case of the universal cover of an Hopf surface which is $\mathbb{C}^2 \setminus \{(0, 0)\}$. The complex manifold $\mathbb{C}^2 \setminus \{(0, 0)\}$ has many holomorphic functions, but it is not holomorphic convex modulo of the pre-image of any subvariety of the Hopf surface.

An explicit motivation for the weakened conjecture can be found in theorem A. This theorem proves the conjecture for projective varieties X satisfying: X has a negative bundle V such that the pullback map identifies a nontrivial extension of \mathcal{O} by V the trivial extension.

Theorem A. *Let X be a projective variety with a negative vector bundle V and $\rho : \tilde{X} \rightarrow X$ its universal covering. If there exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^*s = 0$ then \tilde{X} is holomorphic convex modulo $\rho^{-1}(Z)$, Z is a subvariety of X .*

The nature of the method used to produce holomorphic functions in the proof of the theorem is another motivation for the conjecture. It will be seen in subsection 2.3 that the method gives: 1) very strong and precise holomorphic convexity properties for \tilde{X} ; 2) a subvariety Z of X for which the holomorphic functions on \tilde{X} created by the method must be constant over $\rho^{-1}(Z)$. The natural appearance of the subvariety Z is not one of the method's shortfalls, but rather one of its strengths, since it is the existence of Z that permits the possible existence of infinite chains. If there are configurations of subvarieties of X whose pre-image contain infinite chains, they must be contained in Z . To quickly put in perspective the scope of theorem A, we note that in corollary 2.6 we show that the conditions of theorem A imply that X has a generically large fundamental group, i.e the general fiber Shafarevich map is zero dimensional. We note that the varieties with generically large fundamental group form a natural class of manifolds to consider when studying the Shafarevich conjecture [Ko93]. In particular, all the difficulties of the conjecture are present for this class of manifolds.

In section 3, we deal with the existence of nonconstant holomorphic functions on the universal cover \tilde{X} of a projective variety X . The known paths to the production of holomorphic functions on \tilde{X} involve the construction of closed holomorphic 1-forms or exhaustion functions with plurisubharmonic properties on \tilde{X} . The construction of the desired closed (1,0)-forms or exhaustion functions on \tilde{X} involve the following methods: (a) properties of the fundamental group $\pi_1(X)$ in combination with Hodge theory and non-abelian Hodge theory (see [Si88] and [EKPR03] for the most recent results and references); (b) curvature properties of X (see for example [SiYa77] and [GrWu77]), (c) explicit descriptions of X (see for example [Gu87] and [Na90]). None of these methods are at the moment sufficiently general to provide a nonconstant holomorphic function for the universal cover of an arbitrary projective variety.

Our approach to the existence of holomorphic functions on \tilde{X} is different. We connect the existence of nonconstant holomorphic functions on \tilde{X} with properties of $\rho^* : Vect(X) \rightarrow Vect(\tilde{X})$, the pullback map for vector bundles. To make our point, we give an extreme example where ρ^* identifies many isomorphism classes. Let X be a projective manifold such that the pullback map identifies all isomorphism classes of holomorphic vector bundles on X that are isomorphic as topological bundles. Then in observation 3.1 we show that \tilde{X} must be Stein.

We proceed to obtain information on ρ^* from the absence of nonconstant holomorphic functions on \tilde{X} . To accomplish this goal, we reexamine the method to produce functions employed in theorem A. The point is that to be able to obtain a holomorphic function on

\tilde{X} from a cocycle $s \in H^1(X, V)$ such that $\rho^*s = 0$ we do not need V to be negative. We need, actually, very weak negativity properties on V as will be illustrated in lemma 3.14 and proposition 3.16. These results are used to prove the main theorem of this section describing the pullback map for absolutely stable vector bundles. A vector bundle is absolutely stable if for any coherent subsheaf $\mathcal{F} \subset E$ with $\text{rk}\mathcal{F} < \text{rk} E$ a multiple of the line bundle $(\text{rk} E \det \mathcal{F} - \text{rk} \mathcal{F} \det E)^*$ can be represented by a nonzero effective divisor. In particular, absolutely stable bundles are stable with respect to all polarizations of X . For projective surfaces absolutely stable bundles are exactly the bundles that are stable with respect to all elements in the closure of the polarization cone.

Theorem B. *Let X be a projective manifold whose universal cover has only constant holomorphic functions. Then:*

a) *The pullback map $\rho_0^* : \text{Mod}_0(X) \rightarrow \text{Vect}(\tilde{X})$ is a local embedding ($\text{Mod}_0(X)$ is the moduli space of absolutely stable bundles).*

b) *For any absolutely stable bundle E there are only a finite number of bundles F with $\rho^*E = \rho^*F$.*

c) *Moreover, there is a finite unramified cover $p : X' \rightarrow X$ associated with E of degree $d < \text{rk} E$ with universal covering $\rho' : \tilde{X} \rightarrow X'$. On X' there is a collection of vector bundles $\{E'_i\}_{i=1, \dots, m}$ on X' with $H^0(\tilde{X}, \text{End}_0 \rho'^* E'_i) = 0$ such that $\rho^*F \simeq \rho^*E$ if and only if:*

$$p^*F = E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_1 \otimes \mathcal{O}(\tau_m)$$

The bundles $\mathcal{O}(\tau_i)$ are flat bundles associated with finite linear representations of $\pi_1(X')$ of a fixed rank k with $\text{rk} E | k$.

The theorem imposes strong constraints on the pullback map ρ^* if \tilde{X} is not to have nonconstant holomorphic functions. In particular, it says that the pullback map should be almost an embedding. The authors believe that this imposition on the pullback map ρ^* should not hold for any projective variety. If the authors are correct then theorem B would imply the existence of nonconstant holomorphic functions on the universal cover of a projective variety (to be addressed in the next publication).

An interesting application of our method used to prove theorem A is a new simple proof of the vanishing of the first cohomology group from negative vector bundles whose rank is smaller than the dimension of the base.

We are grateful to L.Katzarkov, T.Pantev and T. Napier for their useful comments. The first author also wants to thank IHES, University of Bayreuth and University of Miami for partial support. The second author also thanks the Centro de Analise, Geometria e Sistemas dinamicos of the IST of Lisbon.

2. CONVEXITY PROPERTIES OF UNIVERSAL COVERS

The first two subsections provide the background and notation for the subsection 2.3 where the weakened Shafarevich conjecture is discussed.

2.1 Holomorphic convexity of universal covers.

A complex manifold X is *holomorphic convex* if for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in X there exists a holomorphic function f on X which is unbounded on the sequence. Shafarevich proposed the following:

Conjecture. (*Shafarevich*) *The universal cover of a projective variety is holomorphic convex.*

By the time the conjecture was formulated, it was known that the holomorphic convexity is a property that is shared by the compact and noncompact universal covers of Riemann surfaces. It was also known that if the fundamental group of a Kahler variety X is abelian then its universal cover \tilde{X} is holomorphic convex. But maybe, the most inspiring evidence was the result of Poincare [Si48] stating that if a bounded domain D in an complex Euclidean space is the universal cover of a projective manifold, then D is holomorphic convex.

The Shafarevich Conjecture predicts that noncompact universal covers of projective manifolds have many holomorphic functions. The holomorphic convexity of the universal cover implies that there is a proper map of \tilde{X} into \mathbb{C}^n . In particular, holomorphic convexity implies that there are enough holomorphic functions to separate points that are not connected by a chain of compact analytic subvarieties. This pointwise holomorphic separability property is the strongest possible for a complex manifold.

We propose a weakened version of holomorphic convexity, that will appear in subsection 2.3 to generalize the Shafarevich conjecture.

Definition 2.1. *Let X be a complex manifold and $\rho : \tilde{X} \rightarrow X$ the universal covering of X . The universal cover X is holomorphic convex modulo an analytic subset $\rho^{-1}(Z)$, $Z \subset X$, if for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ $x_i \in \tilde{X}$ such that $\{\rho(x_i)\}$ has no accumulation points on Z , there exists a holomorphic function f on \tilde{X} which is unbounded on the sequence.*

Both holomorphic convexity and holomorphic convexity modulo an analytic subset of an universal cover imply the abundance of holomorphic functions on \tilde{X} .

There is also a generalization of the notion of holomorphic convexity to line bundles other than \mathcal{O} . Let X be a complex manifold and L a line bundle on X with an Hermitian metric h . X is *holomorphic convex with respect to (L, h)* if for every infinite discrete sequence $\{x_i\}_{i \in \mathbb{N}}$ of points in X there exists a section $s \in H^0(X, L)$ such that the function $|s|_h$ is unbounded on the sequence (holomorphic convexity is the special case of holomorphic convexity with respect to the trivial line bundle equipped with the trivial

metric). In subsection 2.3 we use the following result from [Na90] concerning holomorphic convexity with respect to positive line bundles. Let X be a smooth projective variety and L a positive line bundle on X . If $p \gg 0$ then the universal cover \tilde{X} , $\rho : \tilde{X} \rightarrow X$, is holomorphic convex with respect to (ρ^*L^p, h) , h is any continuous Hermitian metric on L^p .

In the work of Kollar [Ko93] and Campana [Ca94] on the Shafarevich conjecture, it was shown that every projective (Kähler) manifold X has a dominant connected rational (meromorphic) map to a normal variety (analytic space) $Sh(X)$, $sh : X \dashrightarrow Sh(X)$ such that:

- There are countably many closed proper subvarieties $D_i \subset X$ such that for every irreducible $Z \subset X$ with $Z \not\subset \bigcup D_i$, one has: $sh(Z) = \text{point}$ if and only if $\text{im}[\pi_1(\bar{Z}) \rightarrow \pi_1(X)]$ is finite, \bar{Z} is the normalization of Z .

The map $sh : X \dashrightarrow Sh(X)$ is called the *Shafarevich map* and $Sh(X)$ is called the *Shafarevich variety of X* . If the Shafarevich conjecture would hold, then Shafarevich map would be a morphism with the property:

- For every subvariety $Z \subset X$, $sh(Z) = \text{point}$ iff $\text{im}[\pi_1(\bar{Z}) \rightarrow \pi_1(X)]$ is finite, \bar{Z} is the normalization of Z .

In subsection 2.3 we present the weakened Shafarevich conjecture and give a motivation for this conjecture for projective manifolds X with a *generically large fundamental group*, i.e $\dim Sh(X) = \dim X$.

2.2 Affine bundles and the negativity of vector bundles.

We recall a construction of affine bundles associated with extensions of a given vector bundle V . We also describe how the negativity properties of the vector bundle V influence the function theory of the affine bundle.

Let X be a complex manifold and V a vector bundle of rank r on X . We will use the common abuse of notation where V also denotes the sheaf of sections of V . An extension of \mathcal{O} by a vector bundle V is an exact sequence:

$$0 \rightarrow V \rightarrow V_\alpha \rightarrow \mathcal{O} \rightarrow 0 \tag{2.1}$$

There is a 1-1 natural correspondence between cocycles $\alpha \in H^1(X, V)$ and isomorphism classes of extensions of \mathcal{O} by V . The extension (2.1) defines an affine bundle, which consists of the pre-image in V_α of a constant nonzero section of the trivial line bundle \mathcal{O} . This affine bundle is independent of the choice of the constant nonzero section and is denoted by A_α . A cocycle α cohomologous to zero corresponds to the trivial extension $V_\alpha = V \oplus \mathcal{O}$. We have that the affine bundle A_α is a vector bundle if and only if (2.1) splits or equivalently if α is cohomologous to zero. Also recall that an affine bundle is a vector bundle if and only if the affine bundle has a section.

The affine bundle A_α can be described in an alternative way. Let E be a vector bundle of rank r over X , $p : \mathbb{P}(E) \rightarrow X$ be the \mathbb{P}^{r-1} -bundle over X , whose points in the fiber

$\mathbb{P}(E)_x$ are the hyperplanes in the vector space E_x , $x \in X$. Associated to a surjection $E \rightarrow F \rightarrow 0$ of vector bundles there is an inclusion $\mathbb{P}(F) \subset \mathbb{P}(E)$ of projective bundles. The affine bundle A_α is $\mathbb{P}(V_\alpha^*) \setminus \mathbb{P}(V^*)$, where the inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V_\alpha^*)$ comes from (2.1) dualized.

Let V be a vector bundle over a complex manifold X . We recall Grauert's characterization of negativity for vector bundles. The projective bundle $\mathbb{P}(V)$ has a naturally defined line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on it. The line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ is the quotient p^*V/F , where F is the tautological hyperplane bundle over $\mathbb{P}(V)$. The vector bundle V is *Grauert negative* if the line bundle $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ is ample.

The negativity properties of a vector imply complex analytic properties of the total space of the vector bundle and of the associated affine bundles. Recall that a complex manifold X is *q-convex* in the sense of Andreotti-Grauert if there is a C^∞ function $\varphi : X \rightarrow \mathbb{R}$ such that outside a compact subset $K \subset X$:

- (i) The subset $\{x \in X : \varphi(x) < c \text{ with } c < \sup_X \varphi\}$ is relative compact.
- (ii) $\varphi|_{X \setminus K}$ is q-convex i.e. the Levi form $L(\varphi) = \sum_{i,j} \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j} dz_i \otimes d\bar{z}_j$ has at most $q - 1$ non-positive eigenvalues at all $x \in X \setminus K$

In subsection 2.3, we will use the following result: if V is a negative vector bundle then any affine bundle A_α is 1-convex (strongly pseudoconvex). This implies that A_α is holomorphic convex and it has a bimeromorphic morphism onto a Stein space.

We give a brief proof of the result mentioned above since it plays an important role in the proof of theorem A and in any of its generalizations [DeO03]. Let $E \rightarrow F$ be a surjection of vector bundles and $\mathbb{P}(F) \subset \mathbb{P}(E)$ be the respective inclusion of projective bundles. The expression for the normal bundle $N_{\mathbb{P}(F)/\mathbb{P}(E)}$ is:

$$N_{\mathbb{P}(F)/\mathbb{P}(E)} \simeq p^*(E^*/F^*) \otimes \mathcal{O}_{\mathbb{P}(F)}(1) \quad (2.2)$$

where $p : \mathbb{P}(F) \rightarrow X$ is the projection. Consider the surjection $V_\alpha^* \rightarrow V^*$, coming from (2.1) dualized, and its induced inclusion $\mathbb{P}(V^*) \subset \mathbb{P}(V_\alpha^*)$. The normal bundle $N_{\mathbb{P}(V^*)/\mathbb{P}(V_\alpha^*)} \simeq \mathcal{O}_{\mathbb{P}(V^*)}(1)$ is positive since V is negative. This implies that complement $\mathbb{P}(V_\alpha^*) \setminus \mathbb{P}(V^*) = A_s$ is strongly pseudoconvex. As a special case, we have that if the extension V_α^* is trivial, $V_\alpha^* = V^* \oplus \mathcal{O}$, then:

$$\mathbb{P}(V^* \oplus \mathcal{O}) \setminus \mathbb{P}(V^*) \simeq V \quad (2.3)$$

Hence the total space $t(V)$ of V is strongly pseudoconvex.

The following is a method to construct many negative bundles of rank $\geq \dim X$ with nontrivial first cohomology. Let L be a very ample line bundle on X which gives an embedding $X \subset \mathbb{P}^n$. There is a surjective map $h : \mathcal{O}_X^{\oplus n+1} \rightarrow L$ which defines a rank n subbundle $\ker h = F \subset \mathcal{O}_X^{\oplus n+1}$. The extension

$$0 \rightarrow F \otimes L^{-1} \rightarrow \bigoplus_{i=1}^{n+1} L^{-1} \rightarrow \mathcal{O} \rightarrow 0 \quad (2.4)$$

is the pullback of the Euler exact sequence of \mathbb{P}^n to X . The vector bundle $F \otimes L^{-1}$ is a negative bundle, $F \otimes L^{-1} \cong \Omega_{\mathbb{P}^n|X}^1$, and $H^1(X, F \otimes L^{-1}) \neq 0$. Namely there is a nontrivial element $s \in H^1(X, F \otimes L^{-1}) \neq 0$ which corresponds to the above nontrivial extension.

The results presented in this paper spring from the existence of nontrivial cocycles $\alpha \in H^1(X, V)$ that become trivial when pulled back to the universal cover. The following standard result (see [La]) shows that this is only possible for infinite covers.

Lemma 2.2. *Let $f : Y \rightarrow X$ a finite morphism between irreducible normal varieties X and Y and V a vector bundle over X . If $s \in H^1(X, V)$ is nontrivial then $f^*s \in H^1(Y, f^*V)$ is also nontrivial.*

This result will also be used in theorem A to bound the dimension of the compact subvarieties in affine bundles associated with nontrivial cocycles.

2.3 The weakened Shafarevich conjecture.

The Shafarevich conjecture claims that a noncompact universal cover of a projective manifold has many holomorphic functions but it also claims the non-existence of infinite chains. As mentioned in the introduction, the second claim may not hold. In this subsection we describe an approach to obtain information on the algebra of holomorphic functions of universal covers which has natural place for infinite chains. The main result of this section, theorem A, motivates the following weakened conjecture:

Conjecture 2.3. *The universal covering \tilde{X} of a projective manifold X is holomorphically convex modulo the pre-image of a subvariety $Z \subset X$.*

As the Shafarevich conjecture, our conjecture also claims a rich algebra of holomorphic functions but it allows the existence of infinite chains. The infinite chains of compact analytic subvarieties would lie in the pre-image of the subvariety $Z \subset X$ described in the conjecture.

We describe briefly the methodology our approach. Let X be a projective manifold, $\rho : \tilde{X} \rightarrow X$ be the universal covering and $\mathcal{O}_{\tilde{X}}(\tilde{X})$ the algebra of global holomorphic functions of \tilde{X} . We derive properties of $\mathcal{O}_{\tilde{X}}(\tilde{X})$ from the existence of nontrivial extensions of \mathcal{O} by negative vector bundles V which become trivial once pulled back to \tilde{X} . From such a special nontrivial extension we construct a map from the universal cover \tilde{X} to the associated affine bundle. This map is a local embedding. From subsection 2.2, it follows that the negativity of V implies strong analytic geometric properties of the algebra of global holomorphic functions of the affine bundle. We use these analytic geometric properties and the local embedding of \tilde{X} in the associated affine bundle to obtain information on $\mathcal{O}_{\tilde{X}}(\tilde{X})$.

Our approach motivates the conjecture in two levels. First, it gives in theorem A an explicit confirmation of the conjecture for projective manifolds X having a nontrivial

extension of \mathcal{O} by a negative vector bundle V whose pullback to \tilde{X} is trivial. Second, it is the nature of the approach to give very strong holomorphic convexity properties for \tilde{X} but also to give a subvariety Z of X for which all the holomorphic functions on \tilde{X} created by the method must be constant over $\rho^{-1}(Z)$. The subvariety Z is the projection into X of the maximal compact analytic subset of the affine bundle associated with the extension (see the proof of theorem A).

Theorem A. *Let X be a projective manifold with a negative vector bundle V and $\rho : \tilde{X} \rightarrow X$ its universal covering. If there exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^*s = 0$ then \tilde{X} is holomorphic convex modulo $\rho^{-1}(Z)$, Z is a subvariety of X .*

Proof. First, we will identify the subvariety $Z \subset X$ described in the theorem. The nontrivial cocycle $s \in H^1(X, V)$ has an associated strongly pseudoconvex affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ originating from the nonsplit exact sequence:

$$0 \rightarrow \mathcal{O}_X \rightarrow V_s^* \rightarrow V^* \rightarrow 0 \quad (2.5)$$

The strongly pseudoconvex manifold A_s (hence holomorphic convex) has a proper holomorphic map onto a Stein space, $r : A_s \rightarrow St(A_s)$ (the Remmert reduction). Moreover, A_s has a subset M called the maximal compact analytic subset of A_s such that the map $r|_{A_s \setminus M} : A_s \setminus M \rightarrow r(A_s \setminus M)$, is a biholomorphism. The subvariety $Z \subset X$ is $Z = p(M)$.

To obtain holomorphic functions on \tilde{X} we will construct a holomorphic map $g : \tilde{X} \rightarrow A_s$ such that $g(\tilde{X}) \not\subset M$ and pullback the holomorphic functions of A_s to \tilde{X} . The pullback of the exact sequence (2.5) to the universal covering \tilde{X} splits into:

$$0 \rightarrow \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}} \oplus \rho^*V^* \rightarrow \rho^*V^* \rightarrow 0$$

since it is associated with the trivial cocycle $\rho^*s \in H^1(\tilde{X}, \rho^*V)$. As observed in (2.3) $A_{\rho^*s} \equiv \mathbb{P}(\mathcal{O}_{\tilde{X}} \oplus \rho^*V^*) \setminus \mathbb{P}(\rho^*V^*) \simeq \rho^*V$, hence \tilde{X} embeds in A_{ρ^*s} as the zero section of ρ^*V^* . The affine bundle A_{ρ^*s} is the fiber product $A_{\rho^*s} = \tilde{X} \times_X A_s$, denote the projection to the second factor by $\rho' : A_{\rho^*s} \rightarrow A_s$ and the embedding of \tilde{X} in A_{ρ^*s} as the zero section of the vector bundle ρ^*V^* by $s : \tilde{X} \rightarrow A_{\rho^*s}$. The holomorphic map $g : \tilde{X} \rightarrow A_s$ will be the composition $g = \rho' \circ s : \tilde{X} \rightarrow A_s$. The map g is a local biholomorphism between \tilde{X} and $g(\tilde{X})$ hence the condition $g(\tilde{X}) \not\subset M$ will hold if $\dim M < \dim X$.

The maximal compact analytic subset of A_s is of the form $M = \cup_{i=1}^k M_i$, where the M_i are the compact irreducible positive dimensional subvarieties of A_s . The following proposition shows that $\dim M_i < \dim X$.

Proposition 2.5. *Let X be a projective manifold with a vector bundle V and V_s be the extension associated with a nontrivial cocycle $s \in H^1(X, V)$. Then any compact subvariety M of the affine bundle $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ satisfies $\dim M < \dim X$.*

Proof. It is clear that if $M \subset A_s$ is a compact subvariety then $\dim M \leq \dim X$ (The intersection of M with any fiber of the projection map $p : A_s \rightarrow X$ will be at most

0-dimensional). We will show that $p^*s|_M \in H^1(M, p^*V|_M)$ must be trivial and that if $\dim M = \dim X$ then $p^*s|_M$ must be nontrivial. These two results prove the desired strict inequality $\dim M < \dim X$.

The triviality of $p^*s|_M$ follows from the triviality of $p^*s \in H^1(A_s, p^*V)$. The equality $p^*s = 0$ holds if and only if the pullback of the exact sequence (3.1) to A_s splits. The affine bundle A_s is $\mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ hence there is a canonical association between the points $y \in A_s$ with $p(x) = y$ and the hyperplanes of $(V_s^*)_x$ surjecting to $(V^*)_x$. From this association one obtains the canonical subbundle $W \subset p^*V_s^*$ on A_s . The vector bundle W is such that the restriction of the surjection $q : V_s^* \rightarrow V^*$ to W is still a surjection and hence an isomorphism. The splitting of $0 \rightarrow \mathcal{O} \rightarrow p^*V_s^* \rightarrow p^*V \rightarrow 0$ is obtained by inverting $q : W \rightarrow V^*$. If $\dim M = \dim X$ then M has an irreducible component M' such that $p^*s|_{M'} = 0$ and the map $p|_{M'} : M' \rightarrow X$ is finite map. Let $n : \hat{M}' \rightarrow M'$ be the normalization map, then $n \circ p|_{\hat{M}'} : \hat{M}' \rightarrow X$ is finite map between normal varieties hence by the lemma 2.2 $p|_{\hat{M}'}^*s \neq 0$ which is a contradiction. \square

Let $\{x_i\}_{i \in \mathbb{N}}$ be a sequence of points in $\tilde{X} \setminus \rho^{-1}(Z)$ such that $\{\rho(x_i)\}_{i \in \mathbb{N}}$ has no accumulation points on Z . The sequence $\{x_i\}_{i \in \mathbb{N}}$ has a subsequence $\{y_i\}_{i \in \mathbb{N}}$ satisfying $\{\rho(y_i)\}_{i \in \mathbb{N}}$ converges to $a \in X \setminus Z$. Consider the sequence $\{g(y_i)\}_{i \in \mathbb{N}}$ of points in A_s , we have two cases: 1) $\{g(y_i)\}_{i \in \mathbb{N}}$ is a discrete sequence of points of A_s ; 2) $\{g(y_i)\}_{i \in \mathbb{N}}$ has a subsequence converging to a point in $a' \in p^{-1}(a)$. If case 1) holds, since A_s is holomorphic convex it follows that there is a function $f' \in \mathcal{O}_{A_s}(A_s)$ that is unbounded on $\{g(y_i)\}_{i \in \mathbb{N}}$. Hence $f' \circ g$ is the desired unbounded function on $\{x_i\}_{i \in \mathbb{N}}$. We proceed to deal with case 2).

Let L be a positive line bundle on X , Napier's result [Na90] states that for $p \gg 0 \exists s \in H^0(\tilde{X}, \rho^*L^p)$ such that $\{|s(y_i)|_{\rho^*h^p}\}_{i \in \mathbb{N}}$ is unbounded (h is an C^∞ Hermitean metric on L). Let $s' \in H^0(X, L^p)$ be such that $a \notin D = (s')_0$. The meromorphic function $h = \frac{s}{\rho^*s'}$ is holomorphic outside $\rho^{-1}(D)$ and unbounded on $\{y_i\}_{i \in \mathbb{N}}$. Assume the existence of a $q \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ with $\inf\{|q(z)| : z \in g^{-1}(a')\} \neq 0$ and vanishing on $\rho^{-1}(D)$. Then for l sufficiently large $f = hq^l$ would be the desired holomorphic function.

The holomorphic function $q \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ satisfying $q(\rho^{-1}(D)) = 0$ and $q(g^{-1}(a)) = 1$ can be obtained by pulling back, using g , a holomorphic function $q' \in \mathcal{O}_{A_s}(A_s)$ that satisfies $q'(p^{-1}(D)) = 0$ and $q'(a') = 1$. The existence of such q' follows from $r : A_s \rightarrow St(A_s)$ being a proper map, $r(a') \cap r(p^{-1}(D \cup Z)) = \emptyset$ and $St(A_s)$ being Stein. \square

It is important to complement theorem A with an example that shows that the hypothesis of the theorem do not imply that the universal cover \tilde{X} is Stein. The Steinness of \tilde{X} holds if the affine bundle A_s is Stein (corollary 2.7), but otherwise \tilde{X} need not be Stein.

Example: We give an example of a projective variety and a vector bundle satisfying the hypothesis of theorem A but whose universal cover is not Stein. Let X be a nonsingular projective variety whose universal cover \tilde{X} is Stein. Let $\sigma : Y \rightarrow X$ be the blow up of

X at a point $p \in X$ $E = \sigma^{-1}(p) = \mathbb{P}^{n-1}$, $n = \dim X$. If we pullback the exact sequence (2.5) to Y and tensor it with $\mathcal{O}(E)$ we obtain:

$$0 \rightarrow \sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E) \rightarrow \bigoplus^{n+1} \sigma^* L^{-1} \otimes \mathcal{O}(E) \rightarrow \mathcal{O}(E) \rightarrow 0 \quad (3.2)$$

The vector bundle $\sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E)$ on Y is negative since $\sigma^* L^{-1} \otimes \mathcal{O}(E)$ is a negative line bundle on Y . Tensoring a negative vector bundle with a globally generated vector bundle gives a negative vector bundle [Ha66]. The pair Y and $\sigma^*(F \otimes L^{-1}) \otimes \mathcal{O}(E)$ satisfies the conditions of theorem A but \tilde{Y} is clearly not Stein (it contains $\pi_1(X)$ copies of \mathbb{P}^{n-1}).

The existence of a vector bundle V on X satisfying the hypothesis of theorem A does impose conditions on X and \tilde{X} . These are described in the following corollary:

Corollary 2.6. *Let X be a projective manifold with a negative vector bundle V and $\rho : \tilde{X} \rightarrow X$ its universal covering. If it exists a nontrivial cocycle $s \in H^1(X, V)$ such that $\rho^* s = 0$ then:*

- a) X has a generically large fundamental group.
- b) The holomorphic functions on \tilde{X} separate points on $\tilde{X} \setminus \rho^{-1}(Z)$, Z a subvariety of X . There is a finite collection of holomorphic functions such that $(f_1, \dots, f_l) : \tilde{X} \rightarrow \mathbb{C}^l$ is a local embedding of \tilde{X} at every point in $\tilde{X} \setminus \rho^{-1}(Z)$.

Proof. In the proof of theorem A a holomorphic map $g : \tilde{X} \rightarrow A_s$ giving a local embedding of \tilde{X} into A_s was constructed. The affine bundle A_s is strongly pseudoconvex, hence it has the Remmert reduction map $r : A_s \rightarrow St(A_s)$ which is a bimeromorphic morphism. The maximal compact subset M of A_s consists of the union of the positive dimensional fibers of the Remmert reduction. Let Z be the projection of $M \subset A_s$ into X . Let $(f'_1, \dots, f'_l) : St(A_s) \rightarrow \mathbb{C}^l$ be an embedding of the Stein space $St(A_s)$ into an complex Euclidean space. The collection of the functions $f_i = f'_i \circ r \circ g$ give the local embedding in described b) since $g(\tilde{X}) \subsetneq M$.

To establish b) we still need to show that if a and b are two different points in $\tilde{X} \setminus Z$ then there is an $f \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ such that $f(a) \neq f(b)$. The $\bar{\partial}$ -method gives that for any positive line bundle L on X the line bundle $\rho^* L^p$ on \tilde{X} for $p \gg 0$ has a section s with $s(a) = 0$ and $s(b) \neq 0$ [Na90]. Let $s' \in H^0(X, L^p)$ be such that $\rho(b) \notin D = (s')_0$. The meromorphic function $h = \frac{s}{\rho^* s'}$ is holomorphic outside $\rho^{-1}(D)$ and $h(b) \neq 0$. As shown in the proof of theorem 3.1 there is a $q \in \mathcal{O}(\tilde{X})$ vanishing on $\rho^{-1}(D)$ and not vanishing at $\rho^{-1}(b)$. Then for l sufficiently large $f = hq^l$ would be the desired holomorphic function.

The projective variety has a generically large fundamental group since no compact subvariety of \tilde{X} passes through any point in $\tilde{X} \setminus \rho^{-1}(Z)$. □

This corollary defines the natural setting for theorem A. As mentioned in the introduction, projective varieties with generically large fundamental group form a natural class

to test and explore the Shafarevich conjecture. If one wants to obtain a result similar to theorem A for a general X one needs to consider semi-negative vector bundles on X . This will be done in the paper [DeO03]. The extensions of semi-negative vector bundles V associated with a nontrivial $s \in H^1(X, V)$ satisfying $\rho^*s = 0$ exist if the Shafarevich conjecture holds. They can be obtained as the pullback to X by the Shafarevich map of the nontrivial extension given by the Euler sequence associated with an embedding of the Shafarevich variety (the target of the Shafarevich map).

Let us consider the special case where the affine bundle A_s over X is a Stein manifold. The condition that A_s is a Stein manifold can be easily fulfilled in the following examples. Over \mathbb{P}^n we have that the the affine bundle A_ω associated with the extension $0 \rightarrow \Omega_{\mathbb{P}^n}^1 \rightarrow \bigoplus^{n+1} \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0$ is isomorphic to the affine variety:

$$F = \mathbb{P}^n \times \mathbb{P}^{n\vee} \setminus \{(x, h) \in \mathbb{P}^n \times \mathbb{P}^{n\vee} | x \in h\}$$

Let X be a projective variety embedded in \mathbb{P}^n and $A_\omega|_X$ the affine bundle associated with pullback to X of the above extension. The affine bundle $A_\omega|_X$ is a Stein manifold since it is a closed subvariety of F . The following is a corollary of theorem A for the case where A_s is Stein.

Corollary 2.7. *Let X be a projective manifold, V a vector bundle and $s \in H^1(X, V)$. Assume furthermore that $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ is a Stein variety. Let $f : Y \rightarrow X$ be any infinite unramified covering s.t. $f^*s = 0$. Then Y is Stein.*

Proof. Since any non-ramified covering of a Stein space is Stein [5] the assumption that A_s is affine yields that $A_s \times_X Y$ is Stein. On the other hand in the proof of Theorem A we saw that $Y \subset A_s \times_X Y$ is a closed analytic subset and so Y is Stein. \square

Corollary 2.7 suggests that the result of Theorem A may also be applicable to orbicoverings of X . Let us first describe precisely the notion of orbicovering in the case of a complex variety. Let X be a complex variety and $S \subset X$ be a proper analytic subset. Consider for any point $q \in S$ the local fundamental group $\pi_q = \pi_1(U(q) \setminus S)$ where $U(q)$ is a small ball in X centered at q . Let $L \subset \pi_1(X \setminus S)$ be a subgroup with the property that $L \cap \pi_q$ is of finite index in π_q for all $q \in S$. Then the nonramified covering of $X \setminus S$ corresponding to L can be naturally completed into a normal complex variety Y_L with a locally finite and locally compact surjective map $f_L : Y_L \rightarrow X$. The map $f_L : Y_L \rightarrow X$ is called an *orbicovering* of X with a ramification set S . The following holds:

Corollary 2.8. *Let X be a projective manifold, V a vector bundle and $s \in H^1(X, V)$. Assume furthermore that $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ is an affine variety. Let $f : Y \rightarrow X$ be any orbicovering s.t. $f^*s = 0$. Then Y is Stein.*

Proof. Since every orbicovering of a Stein space is also Stein (see Theorem 4.6 of [5]) the proof is exactly the same as the proof of Corollary 3.5. \square

3. THE PULLBACK MAP FOR VECTOR BUNDLES AND HOLOMORPHIC FUNCTIONS ON UNIVERSAL COVERS

The positive results on Shafarevich conjecture generally involve the existence of non-isomorphic vector bundles on X which become isomorphic after the pullback to \tilde{X} . For example, the theorem of L.Katzarkov [4] establishes the holomorphic convexity of \tilde{X} for a projective surface X under the assumption of the existence of an almost faithful linear representation of $\pi_1(X)$. In this case all the bundles on X corresponding to the representations of the same rank of the fundamental group are becoming equal on \tilde{X} . Let X be a projective manifold, $\rho : \tilde{X} \rightarrow X$ its universal covering and $\rho^* : Vect(X) \rightarrow Vect(\tilde{X})$ the pullback map for isomorphism classes of holomorphic vector bundles. This section investigates the relation between the properties of the pullback map ρ^* and the existence of holomorphic functions on \tilde{X} .

We start by considering the case of projective manifolds whose pullback map ρ^* identifies the isomorphism classes that are isomorphic as topological vector bundles. If the pullback satisfy this property, then there are plenty of distinct vector bundles on X whose pullbacks are identified. In particular, any two bundles which can be connected by an analytic deformation are bound to be identified on \tilde{X} . This very rich collection of bundles that are identified via the pullback map imply the following result on the algebra of global holomorphic functions on \tilde{X} holds:

Observation 3.1. *Let X be a projective manifold whose pullback map ρ^* identifies isomorphism classes of holomorphic vector bundles that are in the same topological isomorphism class. Then the universal cover \tilde{X} is Stein.*

Proof. Let X be a subvariety of \mathbb{P}^n . Let $\alpha \in H^1(X, \Omega_{\mathbb{P}^n}^1|_X)$ be the cocycle associated with the extension of $\Omega_{\mathbb{P}^n}^1|_X$ coming from the Euler exact sequence, $(\Omega_{\mathbb{P}^n}^1|_X)_\alpha$. The pullback $\rho^*\alpha = 0$, since $\rho^*(\Omega_{\mathbb{P}^n}^1|_X)_\alpha$ is isomorphic to the pullback $\rho^*(\Omega_{\mathbb{P}^n}^1|_X \oplus \mathcal{O})$, topologically they are the same bundle. The result then follows from corollary 2.7 and the paragraph preceding it. \square

The condition that ρ^* identifies isomorphism classes of holomorphic vector bundles that are in the same topological isomorphism class could be replaced by the following apparently weaker condition: any extensions of a vector bundle V by another vector bundle V' are identified under ρ^* .

This section is mainly concerned with the implications of the absence of nonconstant holomorphic functions on \tilde{X} on the pullback map ρ^* . The condition that \tilde{X} has no non-constant holomorphic functions lies on the opposite side of the conclusion of observation 3.1, stating that \tilde{X} is Stein. We will show that this condition on \tilde{X} has implications that are quite opposite to the assumption of the observation 3.1. More precisely, the absence of nonconstant holomorphic function on \tilde{X} implies that the pullback map ρ^* is almost an imbedding. This conclusion lies in strict contrast with the assumption of

3.1, which implies that the pullback map identifies many bundles. The authors believe that this result should not hold for projective varieties with infinite $\pi_1(X)$; see the remarks at the end of this section. If the authors are correct, our approach will show that there are always nonconstant holomorphic functions on noncompact universal covers \tilde{X} of projective varieties.

Let $\rho^* : \text{Vect}(X) \rightarrow \text{Vect}(\tilde{X})$ be the pullback map sending the set of isomorphism classes of vector bundles on X into the set of isomorphism classes of vector bundles on \tilde{X} . The flat vector bundle on X obtained from a linear representation τ of the fundamental group of X or its sheaf of sections is denoted by $\mathcal{O}(\tau)$. By construction $\rho^*\mathcal{O}(\tau)$ is the trivial bundle on the universal covering \tilde{X} with the rank of τ . It is clear that two bundles $F \otimes \mathcal{O}(\tau)$ and $F \otimes \mathcal{O}(\tau')$ become isomorphic on \tilde{X} for any bundle F if the rank of the representations τ and τ' is the same. The main result of this section states that if \tilde{X} has no holomorphic functions and two bundles E and E' on X have isomorphic pullback on \tilde{X} then $E = F \otimes \mathcal{O}(\tau)$ and $E' = F \otimes \mathcal{O}(\tau')$ for some bundle F .

In order to better understand the map ρ^* and, in particular, to study its local properties, one should put a structure of an analytic scheme into the sets $\text{Vect}(X)$ and $\text{Vect}(\tilde{X})$. The scheme structure for $\text{Vect}(X)$ for X projective is well understood. Below, we recall the key facts that are relevant to our goals. The analytic scheme structure theory for $\text{Vect}(\tilde{X})$ is less understood. We would like to note that it is not our interest to develop such a theory in this paper. The only facts that we use from $\text{Vect}(\tilde{X})$ are that distinct points correspond to non-isomorphic bundles and that the formal tangent space at a vector bundle E on \tilde{X} exists and it is equal to $H^1(\tilde{X}, \text{End}E)$.

3.1 Stability Background.

To obtain a good parameterizing scheme for vector bundles on a projective variety X we have to consider some stability conditions on the bundles, see below. There is an algebraic parameterization for H -stable bundles with given topological invariants. This parameterization space has all the basic properties of a coarse moduli space (see for example [HuLe97], [Ma77]).

Let E be a vector bundle on a projective variety X of dimension n and H be an arbitrary element in the closure of the polarization cone $P \subset H^{1,1}(X, \mathbb{R})$. E is said to be H -semistable if the inequality $(\text{rk } E \det \mathcal{F} - \text{rk } \mathcal{F} \det E) \cdot H^{n-1} \leq 0$ holds for all coherent subsheaves $\mathcal{F} \subset E$. Moreover, if for all coherent subsheaves $\mathcal{F} \subset E$ of lower rank $(\text{rk } E \det \mathcal{F} - \text{rk } \mathcal{F} \det E) \cdot H^{n-1} < 0$ holds then E is said to be H -stable. The vector bundle E is H -unstable if it has an H -destabilizing subsheaf \mathcal{F} , i.e there is a coherent subsheaf \mathcal{F} with $0 < \text{rk } \mathcal{F} < \text{rk } E$ such that $(\text{rk } E \det \mathcal{F} - \text{rk } \mathcal{F} \det E) \cdot H^{n-1} > 0$ holds. The number $\mu_H(\mathcal{F}) = (\det \mathcal{F} / \text{rk } \mathcal{F}) \cdot H^{n-1}$ is called the H -slope of \mathcal{F} . H -stability of E is equivalent to the fact that any coherent subsheaf of E with smaller rank has a smaller H -slope than E . The notion of H -stability for H is the same as for aH , $a \in \mathbb{R}$ and $a > 0$.

Denote by $P(X)$ the polarization cone of X in the real space $H^{1,1}(X, \mathbb{R})$ and denote its closure by $\bar{P}(X)$. Since the base of the closure $\mathbb{P}(\bar{P}(X))$ of the cone $\bar{P}(X)$ in real

projective space $\mathbb{P}(H^{1,1}(X, \mathbb{R}))$ is compact the notion of a stable bundle with respect to the closure of the polarization cone is well defined. In particular we have the following result (see [Bo78], [Bo94] and [HuLe97]).

Definition 3.2. *The cone of effective divisors on X , denoted by K_{eff} , is the cone in the group $\text{Pic}(X) \otimes \mathbb{R}$ generated using only non-negative real coefficients by the representatives of effective divisors. The cone of effective divisors without the zero, $K_{eff} \setminus 0$, will be denoted by K_{eff+} . Similarly the cone of anti-effective divisors is denoted by $-K_{eff}$ and $-K_{eff+}$.*

Lemma 3.3. *Let E be a vector bundle over projective surface X which is stable with respect to all elements H in $\bar{P}(X)$. Then any coherent proper subsheaf $\mathcal{F} \subset E$ satisfies $rk\mathcal{F}detE - rkE det\mathcal{F} \in K_{eff+}$.*

Proof. Thanks to the stability assumption we have that $(rk\mathcal{F}detE - rkE det\mathcal{F}) \cdot H > 0$ for any H in the closure of the polarization cone. Since the cone $\mathbb{P}(\bar{P}(X))$ is compact, it is also true in the neighborhood of the cone. Thus, using Kleiman duality for surfaces, we obtain that a positive multiple of $(rk\mathcal{F}detE - rkE det\mathcal{F})$ is effective and nonzero.

For a general projective variety X the stability property is captured on surfaces which are complete intersections in the initial variety. Thus we make the following definition (see also [Bo78])

Definition 3.4. *A vector bundle E is absolutely stable if for any coherent subsheaf $\mathcal{F} \subset E$ with $rk\mathcal{F} < rkE$ the following holds: $rkE det\mathcal{F} - rk\mathcal{F} detE$ belongs to the cone $-K_{eff+}$. In particular an absolutely stable bundle is stable with respect to all polarizations.*

The condition of absolute stability is the right condition for the formulation of our results later in this section. We will need some properties of the bundle $\text{End}E$ for an H -stable bundle E . We will also need results on the theory of stable bundles for smooth projective curves and on how stability behaves under restriction maps. We start with the basic lemma (for a proof see, for example, chapter 1 of [HuLe97]):

Lemma 3.5. *Let E be a vector bundle on X which is stable with respect to some $H \in P(X)$. Then $H^0(X, \text{End}_0E) = 0$.*

If C is a smooth curve and E is a stable vector bundle over C then by a classical result of Narasimhan-Seshadri End_0E is obtained from a unitary representation τ of the fundamental group $\pi_1(C)$ in $PSU(n)$, $n = rkE$. The elements of $PSU(n)$ act on the matrices in $\text{End}\mathbb{C}^n$ by conjugation. Since the bundle E is stable the representation τ is irreducible.

Lemma 3.6. *If E is a stable vector bundle over a smooth projective curve C then the bundle $\text{End}E$ is a direct sum of stable vector bundles of degree 0.*

Proof. This fact is well known and the decomposition into a direct sum of stable bundles corresponds to the decomposition of the unitary representation of $\pi_1(X)$ in $PSU(n) \subset SU(n^2 - 1)$ under the above imbedding. \square

Let E be a vector bundle over a projective variety X . Let $C_E \subset \text{Pic}X \otimes \mathbb{R}$ be the cone generated by the classes $\det \mathcal{L}$ for all coherent subsheaves $\mathcal{L} \subset \bigcup_{n=0}^{\infty} (\text{End}E)^{\otimes n}$ with $\text{rk } \mathcal{L} = 1$. Let us remember the following result from [Bo94] which follows from invariant theory:

Lemma 3.7. *The cone C_E is also generated by the elements of the form: $\det \mathcal{F} - (\text{rk} \mathcal{F}_i / \text{rk} E) \det E$, where $\mathcal{F} \subset E$ is a proper coherent subsheaf of E and some elements in $-K_{\text{eff}}$. For any bundle E of rank k there is a natural reductive structure group $G_E \subset GL(k)$ of E such that C_E is generated by the line subbundles L corresponding to the characters of parabolic subgroups in G_E .*

The group G_E is defined modulo scalars by the set of subbundles $L \in (\text{End}E)^{\otimes n}$ for all n with $c_1(L) = 0$. If $G_E = GL(k), SL(k)$ then the line subbundles L are exactly the line bundles $\det \mathcal{F} - (\text{rk} \mathcal{F}_i / \text{rk} E) \det E$. However, if the group G_E is smaller than above then the line bundles generating C_E correspond to determinants of special subsheaves of E .

Corollary 3.8. *Let E be an absolutely stable vector bundle on a smooth projective variety X and $\mathcal{A} \subset \text{End}E$ be a coherent subsheaf. Then $\det \mathcal{A} \in -K_{\text{eff}}$. If E is absolutely stable and $E = E' \otimes F$ then both E', F are absolutely stable since the corresponding parabolic group G_E is contained in the group product $G_{E'} \times G_F$ and the cone C_E is a sum $C_{E'} + C_F$.*

Proof. Since $\det \mathcal{A} \subset (\text{End}E)^{\otimes \text{rk} \text{End}E}$, $\det \mathcal{A} \in C_E$ and the previous lemma implies that $\det \mathcal{A} = \sum a_i (\det \mathcal{F}_i - (\text{rk} \mathcal{F}_i / \text{rk} E) \det E)$ where $a_i \geq 0$ and \mathcal{F}_i coherent subsheaves of E . The conclusion follows from the condition of absolute stability, i.e all elements $\det \mathcal{F}_i - \det E (\text{rk} \mathcal{F}_i / \text{rk} E)$ belong to $-K_{\text{eff}}$. The parabolic subgroups in $G_{E'} \times G_F$ are products of the parabolic subgroups in $G_{E'}, G_F$ which implies the result. \square

Many properties of H -stable bundles on arbitrary projective varieties can be derived from their restrictions on smooth curves. As is manifested in the following two results.

Lemma 3.9. *Let X be a projective variety, H a polarization of X , E an H -stable vector bundle and C a generic curve in kH^{n-1} for $k \gg 0$. Then:*

- 1) *The restriction of E to C is stable.*
- 2) *Any saturated coherent subsheaf $\mathcal{F} \subset \text{End}E|_C$ with $\mu_H(\mathcal{F}) = 0$ is a direct summand of $\text{End}E|_C$.*
- 3) *The set of saturated subsheaves \mathcal{F} of $\text{End}E$ with $\mu_H(\mathcal{F}) = 0$ coincide via the restriction map with the similar set for $\text{End}E|_C$ on C .*
- 4) *The bundle $\text{End}E$ is H -semistable and it is a direct sum of H -stable bundles F_i with $\mu_H(F_i) = 0$.*

Proof. 1), 2) follows from general results, see for example [Bo78], [Bo94] and [HuLe97]. 3) is a consequence of the following result: the algebras $\bigoplus_{i=1}^{\infty} H^0(X, S^i \text{End}_0 E)$ and $\bigoplus_{i=1}^{\infty} H^0(C, S^i \text{End}_0 E|_C)$ are isomorphic up to a level l depending on C (depending on k). This isomorphism follows from the vanishing of the cohomology of coherent sheaves

on projective varieties after being tensored with a sufficient large multiple of an ample line bundle. In particular, as a consequence of the result, we obtain that the algebra $H^0(X, \text{End}(\text{End}E))$ coincides with $H^0(C, \text{End}(\text{End}E|_C))$. This implies that the direct summands of $\text{End}E|_C$ are the restrictions to C of the direct summands of $\text{End}E$. Hence 3) follows if every saturated subsheaf \mathcal{F} of $\text{End}E$ with $\mu_H(\mathcal{F}) = 0$ is a direct summand of $\text{End}E$. This last statement is a consequence of 2). 2) implies that $\mathcal{F}|_C$ is a direct summand of $\text{End}E|_C$ and therefore, by the above, \mathcal{F} is a direct summand of $\text{End}E$. 4) follows from 3) and lemma 3.6. \square

Corollary 3.10. *Let E be an absolute stable vector bundle on a projective variety X . Then $\text{End}E = \bigoplus_{i=1}^l F_i$, where the F_i are absolute stable bundles with $\mu_H(F_i) = 0$ for any H in $P(X)$.*

Proof. The vector bundle E is H -stable for all polarizations H of X . Fix a polarization H , lemma 3.9 4) implies that $\text{End}E = \bigoplus_{i=1}^l F_i$ where all the F_i are H -stable with $\mu_H(F_i) = 0$. Our claim is that the F_i are absolute stable vector bundles.

Let \mathcal{F} be a coherent subsheaf of one of the direct summands F_i with $\text{rk}\mathcal{F} < \text{rk}F_i$. We need to show that $\det\mathcal{F} \in -K_{\text{eff}+}$. Lemma 3.8 almost gives the result, $\det\mathcal{F} \in -K_{\text{eff}}$. If $\det\mathcal{F} \notin -K_{\text{eff}+}$ then $\mu_H(\mathcal{F}) = 0$. Lemma 3.9 3) implies that \mathcal{F} is a direct summand of $\text{End}E$ and hence also of F_i . This is not possible since F_i is H -stable. \square

3.2 Holomorphic functions and flat bundles.

In section 2, we described a method to obtain holomorphic functions on the universal cover $\rho : \tilde{X} \rightarrow X$ of a complex manifold X . The method involved negative vector bundles E with a nontrivial cocycle $\alpha \in H^1(X, E)$ satisfying $\rho^*\alpha = 0$. In this subsection, we re-examine the method to be able to apply it to vector bundles with very weak negativity properties, see lemma 3.14 and proposition 3.16. One interesting characteristic of these two results is that: if the vector bundle E satisfies the weak negativity conditions described in the results, the method fails to give nonconstant holomorphic functions on \tilde{X} only if V is also a flat bundle. This is interesting because Hodge theory and nonabelian Hodge theory obtain holomorphic functions from flat bundles. Later in the subsection 3.3, we will rely on this seemingly contradictory role of flat bundles to describe the pullback map ρ^* . We will visit the production of holomorphic functions on the universal covers of Kahler manifolds involving the existence of flat bundles associated with infinite linear representations of $\pi_1(X)$. We give simple proofs for some special cases. The strongest result in this direction follows from [EKPR03] and is described in observation 3.20.

As previously announced, we now reexamine the method of producing holomorphic functions developed in section 2. The goal is to be able use the main idea of the method

to get functions on \tilde{X} for the weakest possible "negativity" assumptions on the vector bundle V .

Definition 3.11. *A sheaf \mathcal{F} on X is universally (generically) globally generated if the sheaf $\rho^*\mathcal{F}$ on the universal cover $\rho : \tilde{X} \rightarrow X$ is (generically) globally generated.*

Lemma 3.12. *Let $p : X' \rightarrow X$ be an infinite unramified Galois covering of a complex manifold X and V a vector bundle over X . If the kernel $p^* : H^1(X, V) \rightarrow H^1(X', p^*V)$ is nontrivial then the vector bundle p^*V on X' has nonzero sections.*

Proof. Let G be the Galois group of the covering and $s \in H^1(X, V)$ be such that $p^*s = 0$. The affine bundle A_{p^*s} is isomorphic to p^*V , but this isomorphism is not G -equivariant with respect to the G -action on p^*V whose quotient is V . More precisely, there are two distinct actions of G on p^*V which differ by affine transformations on p^*V . One of the actions has as the quotient space the vector bundle V and the other the affine bundle A_s . The action whose quotient is A_s can not preserve the zero section of p^*V and hence p^*V has nontrivial sections. \square

Lemma 3.13. *Let V be a vector bundle with a nontrivial cocycle $\alpha \in H^1(X, V)$ such that $\rho^*\alpha = 0$. Then there is an universally globally generated coherent subsheaf $\mathcal{F} \subset V$ such that the cocycle α comes from a cocycle $\beta \in H^1(X, \mathcal{F})$.*

Proof. It follows from lemma 3.12 that vector bundle ρ^*V has nontrivial sections. Let \mathcal{F} be the subsheaf of V whose stalk at $x \in X$ consists of the germs of the global sections of ρ^*V at one pre-image $\tilde{x} \in \rho^{-1}$. Any choice of pre-image would give the same stalk since $\pi_1(X)$ acts on ρ^*V and on $H^0(\tilde{X}, \rho^*V)$ as well. The sheaf \mathcal{F} is coherent because of the strong noetherian property of coherent sheaves on complex manifolds. By construction the sheaf \mathcal{F} is universally globally generated.

Let $i_* : H^1(X, \mathcal{F}) \rightarrow H^1(X, V)$ and $q_* : H^1(X, V) \rightarrow H^1(X, V/\mathcal{F})$ be the morphisms from the cohomology long exact sequence associated with $0 \rightarrow \mathcal{F} \rightarrow V \rightarrow V/\mathcal{F} \rightarrow 0$. The existence a cocycle $\beta \in H^1(X, \mathcal{F})$ with $\alpha = i_*\beta$ follows if $q_*\alpha = 0$. The extension $0 \rightarrow V \rightarrow V_\alpha \rightarrow \mathcal{O} \rightarrow 0$ associated with the cocycle α induces the exact sequence:

$$0 \rightarrow V/\mathcal{F} \rightarrow V_\alpha/\mathcal{F} \rightarrow \mathcal{O} \rightarrow 0 \quad (3.1)$$

The triviality of $q_*\alpha$ holds if (3.1) splits. The exact sequence (3.1) is the quotient of the the exact sequence:

$$0 \rightarrow \rho^*V/\rho^*\mathcal{F} \rightarrow (\rho^*V)_{\rho^*\alpha}/\rho^*\mathcal{F} \rightarrow \mathcal{O} \rightarrow 0 \quad (3.2)$$

via the action of $\Gamma = \pi_1(X)$ on $(\rho^*V)_{\rho^*\alpha}$ that gives V_α . The extension of ρ^*V associated with $\rho^*\alpha$ splits by the hypothesis, but this splitting is not $\pi_1(X)$ -invariant. The splitting is given by a section $s \in H^0(\tilde{X}, \rho^*V_\alpha)$, that is not preserved by the $\pi_1(X)$ -action. On

the other hand, this splitting induces a Γ -invariant splitting of (3.2) since $s - \gamma s \in H^0(\tilde{X}, \rho^* \mathcal{F})$ and $\rho^* \mathcal{F} / \Gamma = \mathcal{F}$.

The next lemma is a flexible tool to produce holomorphic functions on the universal coverings that will be a key ingredient of our results.

Lemma 3.14. *Let \mathcal{F} be a universally generically globally generated coherent torsion free sheaf on a complex manifold X such that $\det(\mathcal{F})^{-k}$ has a nontrivial section. Then one of the following holds:*

- 1) \tilde{X} has a nonconstant holomorphic function.
- 2) \mathcal{F} is the sheaf of sections of a flat bundle, $\mathcal{F} \cong \mathcal{O}(\tau)$.

Proof. Let s_1, \dots, s_r be a collection of sections of $\rho^* \mathcal{F}$ generating $\rho^* \mathcal{F}$ generically, where r is the rank of \mathcal{F} . From the sections s_1, \dots, s_r one gets a nontrivial section of $\det \rho^* \mathcal{F}$ $s = s_1 \wedge \dots \wedge s_r \in H^0(\tilde{X}, \det(\rho^* \mathcal{F}))$, the pairing of $s^{\otimes n}$ with a nontrivial section $t \in H^0(\tilde{X}, \det(\rho^* \mathcal{F})^{-n})$ gives a holomorphic function f on \tilde{X} . By hypothesis the function f is nonzero on a open set of \tilde{X} . The function $f(p) = 0$ at $p \in \tilde{X}$ in case: i) $s_1(p), \dots, s_r(p)$ do not generate $\rho^* \mathcal{F}_p / m_p \rho^* \mathcal{F}_p$ or ii) $t(p)$ is zero.

Suppose statement 1) does not hold. Then f must be a nonzero constant function, which implies that $s_1(p), \dots, s_r(p)$ are linear independent at all $p \in \tilde{X}$. Hence the morphism $(s_1, \dots, s_r) : \mathcal{O}^r \rightarrow \rho^* \mathcal{F}$ induced by the sections is an isomorphism. The nonexistence of holomorphic functions on \tilde{X} implies that all sections of $\rho^* \mathcal{F} = \mathcal{O}^r$ are constant. The linear action of $\pi_1(X)$ on $H^0(\tilde{X}, \mathcal{O}^r) = \mathbb{C}^r$ gives a representation $\tau : \pi_1(X) \rightarrow GL(r, \mathbb{C})$ and \mathcal{F} is the sheaf of sections of the flat vector bundle $\tilde{X} \times_{\tau} \mathbb{C}^r$. \square

Corollary 3.15. *Let X be a projective variety such that $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. If E is a vector bundle such that $\det E^* \in K_{eff}$ then E is not universally generically global generated unless $\det E$ has finite order in $\text{Pic} X$ and E is flat.*

The next result is an application of lemma 3.14 for vector bundles.

Proposition 3.16. *Let E be an absolutely stable vector bundle over a projective manifold X with $\det E = \mathcal{O}$. If there is a nontrivial cocycle $\alpha \in H^1(X, E)$ such that $\rho^* \alpha = 0$ then one of the following possibilities holds:*

- 1) \tilde{X} has nonconstant holomorphic functions.
- 2) $E \cong \mathcal{O}(\tau)$ and α is contained in the image of $H^1(X, \mathbb{C}(\tau))$ in $H^1(X, E)$.

Proof. Lemma 3.13 states that there is a nontrivial universally globally generated coherent subsheaf \mathcal{F} of E such that α is contained in the image of $H^1(X, \mathcal{F})$ in $H^1(X, E)$. If $\text{rk} \mathcal{F} < \text{rk} E$ then the absolute stability of E would imply that the line bundle $(\det \mathcal{F})^* \in K_{eff+}$. Hence \mathcal{F} is not a flat vector bundle and by lemma 3.14 \tilde{X} must have nonconstant holomorphic functions.

If $\text{rk} \mathcal{F} = \text{rk} E$ then E is universally generically globally generated. Since \tilde{X} has no nonconstant holomorphic functions, lemma 3.14 gives that E is a flat vector bundle $\tilde{X} \times_{\tau} \mathbb{C}^r$, for some representation $\tau : \pi_1(X) \rightarrow GL(r, \mathbb{C})$. Notice that for $\alpha \in H^1(X, E)$

with $\rho^*\alpha = 0$ the section $s \in H^0(\tilde{X}, \mathcal{O}^r)$ ($\rho^*E \cong \mathcal{O}^r$) with $ds = \rho^*\alpha$ is constant. Hence we obtain that s belongs to the image of $s' \in H^1(X, \mathbb{C}(\tau))$ under a natural map $H^1(X, \mathbb{C}(\tau)) \rightarrow H^1(X, E)$. \square

Both lemma 3.14 and proposition 3.16 give a method to obtain holomorphic functions on \tilde{X} from vector bundles that are not flat. The results that follow give a description of how flat bundles can produce holomorphic functions on the universal cover. Recall the notation described in the introduction to this section, a representation $\tau : \pi_1(X) \rightarrow GL(m, \mathbb{C})$ defines the flat vector bundle $\mathcal{O}(\tau)$ on X . We denote the sheaf of sections of $\mathcal{O}(\tau)$ also by $\mathcal{O}(\tau)$ and the subsheaf of locally constant sections by $\mathbb{C}(\tau)$. The imbedding $\mathbb{C}(\tau) \hookrightarrow \mathcal{O}(\tau)$ induces a map of the cohomology groups.

Proposition 3.17. *Let X be a complex manifold and $\tau : \pi_1(X) \rightarrow GL(m, \mathbb{C})$ be a representation of $\pi_1(X)$. If \tilde{X} has no nonconstant holomorphic functions then the map $H^1(X, \mathbb{C}(\tau)) \rightarrow H^1(X, \mathcal{O}(\tau))$ is an imbedding.*

Proof. Consider the exact sequence of sheaves on X associated with the differential d :

$$0 \longrightarrow \mathbb{C}(\tau) \longrightarrow \mathcal{O}(\tau) \xrightarrow{d} d\mathcal{O}(\tau) \longrightarrow 0$$

where $d\mathcal{O}(\tau)$ is the image subsheaf in $\Omega^1(X) \otimes \mathcal{O}(\tau)$. From the long cohomology exact sequence, we have:

$$H^0(X, d\mathcal{O}(\tau)) \rightarrow H^1(X, \mathbb{C}(\tau)) \rightarrow H^1(X, \mathcal{O}(\tau))$$

If the second map has a nontrivial kernel, then $H^0(X, d\mathcal{O}(\tau))$ is nonzero. Any section of $H^0(X, d\mathcal{O}(\tau))$ induces a closed holomorphic (1,0)-form on \tilde{X} with values in \mathcal{O}^n and by integration a set of nonconstant holomorphic functions. \square

Corollary 3.18. *Let X be a complex manifold and $\tau : \pi_1(X) \rightarrow GL(m, \mathbb{C})$ be a representation of $\pi_1(X)$. If $h^1(X, \mathbb{C}(\tau)) > h^1(X, \mathcal{O}(\tau))$ then \tilde{X} has nonconstant holomorphic functions.*

Proof. Suppose the corollary did not hold then $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. Hence $H^0(X, \mathbb{C}(\tau)) = H^0(X, \mathcal{O}(\tau))$ since any holomorphic section of $\rho^*\mathcal{O}(\tau)$ on \tilde{X} is constant. Apply proposition 3.17 to get the contradiction. \square

Proposition 3.19. *Let X be a complex manifold such that \tilde{X} has no nonconstant holomorphic functions. Then the following properties hold:*

1) *For any bundle E there is at most one representation τ , up to conjugation, such that $E = \mathcal{O}(\tau)$.*

2) If X is Kahler then $H^1(X, \mathbb{C}(\tau)) = H^1(X, \mathcal{O}(\tau)) = 0$ for any unitary representation τ of $\pi_1(X)$. In particular, $H^1(X, \mathcal{O}) = 0$, $\text{Pic}_0 X \otimes \mathbb{Q} = 0$ and $\text{Pic} X \otimes \mathbb{Q} = \text{NS}(X) \otimes \mathbb{Q}$.

Proof. 1) The structure of a bundle $\mathcal{O}(\tau)$ on E is the same as a flat connection on E . Suppose there were two different structures $\mathcal{O}(\tau)$ and $\mathcal{O}(\tau')$ on E , they would induce two flat $(1,0)$ -connections whose difference is a non-zero element of $H^0(X, d\text{End}\mathcal{O}(\tau))$. The desired conclusion follows from the argument in proof of proposition 3.17.

2) Notice that for a unitary representation τ the cohomology of $\mathbb{C}(\tau)$ and $\mathcal{O}(\tau)$ satisfy the Hodge decomposition. In particular, there is an isomorphism of vector spaces $H^1(X, \mathbb{C}(\tau)) = H^1(X, \mathcal{O}(\tau)) \oplus H^0(X, \Omega^1 \otimes \mathcal{O}(\bar{\tau}))$. Corollary 3.18 implies that $H^0(X, \Omega^1 \otimes \mathcal{O}(\bar{\tau})) = 0$ and by the Hodge conjugation isomorphism, it follows that $H^1(X, \mathcal{O}(\tau)) = 0$. As a special case, we obtain $H^1(X, \mathcal{O}) = 0$. □

Observation 3.20. *The unpublished paper [EKPR03] has implicit the following consequence: if a smooth Kahler variety X has an infinite linear representation of the fundamental group then its universal cover has nonconstant holomorphic functions. For projective surfaces this result appears in [Ka97]. For Kahler manifolds, our three last results follow this consequence. On the other hand, the proof of our results is more direct and significantly simpler.*

Remark: The method of [EKPR03] using non-abelian Hodge theory to construct holomorphic functions on the universal covers \tilde{X} requires that the base manifold X is Kahler. If X is not Kahler the properties of the fundamental group of X can not guarantee the existence of non-constant holomorphic functions on \tilde{X} . This follows from the results of Taubes on anti-self-dual structures on real 4-manifolds [Ta92]. Taubes showed that every finitely presented group is the fundamental group of a compact complex 3-fold X that has a foliation by \mathbb{P}^1 with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This in turn implies that the universal cover \tilde{X} has no non-constant holomorphic functions. The universal cover \tilde{X} also has a foliation by \mathbb{P}^1 with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Any one of these \mathbb{P}^1 has a 2-concave and hence pseudoconcave neighborhood since their normal bundle is Griffiths-positive [Sc73]. The conclusion follows since a complex manifold with a pseudoconcave open subset has only constant holomorphic functions. Moreover the variety \tilde{X} with X being a twistor space for a sufficiently generic anti-self-dual metric on the underlying 4-dimensional variety has no meromorphic functions. Indeed the field of meromorphic functions on \tilde{X} is always a subfield of the field of meromorphic functions in the normal neighborhood of \mathbb{P}^1 and the latter is always a subfield of $C(x, y)$ and consists of constant functions only for a generic neighborhood of \mathbb{P}^1 with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

3.3 Pullback map for line bundles.

We describe the implications of the absence of nonconstant holomorphic functions on the universal cover \tilde{X} on the pullback map for line bundles $\rho^* : \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$.

Definition 3.21. *The cone of divisors on X generated by the divisors which become effective on \tilde{X} is denoted by \tilde{K}_{eff} . Following 3.2, \tilde{K}_{eff+} is also defined.*

If \tilde{X} has no non-constant holomorphic functions then the cone \tilde{K}_{eff} contains K_{eff} but does not contain any elements from $-K_{eff+}$. Suppose $-K_{eff+} \cap \tilde{K}_{eff} \neq \emptyset$ then there is an divisor effective D of X such that both line bundles $\rho^*\mathcal{O}(D)$ and $\rho^*\mathcal{O}(-D)$ have nontrivial sections. The pairing of these sections gives a non-constant holomorphic function. In particular, the image of $\rho^* : \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$ is nontrivial and there are the following possibilities:

P1. The cone \tilde{K}_{eff+} is separated by a hyperplane L from $-K_{eff+}$.

P1'. The cone \tilde{K}_{eff} coincides with K_{eff} . This is a special case of P1. In particular, it holds if $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ and $\text{Pic}(X) = \mathbb{Z}$.

P2. The closure of the cone \tilde{K}_{eff} in $\text{Pic}(X) \otimes \mathbb{R}$ intersects with the closure of $-K_{eff}$ outside of 0.

The following result describes the kernel of $\rho^* : \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$ for Kahler manifolds whose universal cover has no nonconstant holomorphic functions.

Proposition 3.22. *Let X be a Kahler manifold such that $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. Then the kernel of the pullback map $\rho^* : \text{Pic}X \rightarrow \text{Pic}\tilde{X}$ is finite and its elements correspond to flat bundles associated with finite characters.*

Proof. Let L be a line bundle in the kernel of ρ^* and let $i : \rho^*L \rightarrow \mathcal{O}_{\tilde{X}}$ be an isomorphism with the trivial line bundle. The isomorphism i is not equivariant with respect to the natural $\pi_1(X)$ -actions on $\mathcal{O}_{\tilde{X}}$ and on ρ^*L giving respectively \mathcal{O}_X and L on X . Hence there is a $g \in \pi_1(X)$ such that the map $(gi)i^{-1} \neq \text{Id} : \mathcal{O}_{\tilde{X}} \rightarrow \mathcal{O}_{\tilde{X}}$. If $(gi)i^{-1} \neq c\text{Id}$ for some constant c then we obtained a nonconstant holomorphic function on \tilde{X} , which can not happen.

Therefore, we have an association of elements of $\pi_1(X)$ with nonzero constants. This association defines a representation $\pi_1(X) \rightarrow \mathbb{C}^*$ and this representation has to be finite, since $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ implies that $H^1(X, \mathbb{C})$ vanishes. The line bundles on the kernel of ρ^* are uniquely determined by the representations described above. Thus, $\text{Ker}(\rho^*)$ is dual to $\pi_1(X)^{ab}$ which is a finite group. \square

3.4 Pullback map for vector bundles.

Assuming that \tilde{X} has no nonconstant holomorphic functions, we use the previous results to describe the pullback map on the moduli spaces of absolutely stable vector bundles on X . Our results are mostly for the spaces of absolutely stable bundles but they have a generalization for the spaces of H -stable bundles, if extra conditions on \tilde{K}_{eff} are added.

In order to describe the local behavior of the pullback map, it is necessary to recall some facts from the theory of deformations of a given vector bundle E on X . The deformation of a vector bundle E over an arbitrary variety splits into the deformation of the projective bundle $\mathbb{P}(E)$ plus a deformation of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. The deformations of $\mathbb{P}(E)$ in the case of a smooth X are parameterized by an analytic subset $B_E \subset H^1(X, \text{End}E)$, $0 \in B_E$ with the action of the group of relative analytic automorphisms $\text{Aut}(E)$ of the bundle $\mathbb{P}(E)$ on B_E . The latter is induced from the natural linear action $\text{Aut}(E)$ on $H^1(X, \text{End}E)$ (with adjoint fiberwise action of $PGL(n)$ on the fiber of the bundle $\text{End}E$). Thus, non-isomorphic bundles (with respect to identical automorphism on X) in the local neighborhood of E are parameterized by the orbits of the group $\text{Aut}E$ with Lie algebra $H^0(X, \text{End}_0(E))$ in $H^1(X, \text{End}E)$.

The space $H^1(X, \text{End}E)$ plays a role of the formal tangent space $T_0(B_E)$ at the point $0 \in B_E$. Natural splitting $\text{End}E = \text{End}_0E \oplus \mathcal{O}$ induces a splitting $H^1(X, \text{End}E) = H^1(X, \text{End}_0E) \oplus H^1(X, \mathcal{O})$. The local deformation scheme of E maps onto a local deformation scheme of $\mathbb{P}(E)$ with a fiber which is locally isomorphic to $H^1(X, \mathcal{O})$. $H^1(X, \mathcal{O})$ parameterizes the (non-obstructed) deformation scheme of line bundles $\mathcal{O}_{\mathbb{P}(E)}(1)$ in $\text{Pic}(\mathbb{P}(E))$ over the deformation scheme of $\mathbb{P}(E)$ which is generically obstructed.

Let $p : \mathcal{E} \rightarrow \Delta$ be an analytic family, over the disc Δ , of vector bundles on X with $E_t = p^{-1}(t)$ as its members. The family E_t gives a deformation of $E = E_0$ and has associated with it a 1st-order deformation cocycle $s \in H^1(X, \text{End}E)$.

Lemma 3.23. *Let $p : \mathcal{E} \rightarrow \Delta$ be family of vector bundles on X that is nontrivial at $t = 0$. If the pullback family $\tilde{p} : \rho^*\mathcal{E} \rightarrow \Delta$ is locally trivial then the kernel of $\rho^* : H^1(X, \text{End}E) \rightarrow H^1(\tilde{X}, \text{End}\rho^*E)$ is nontrivial.*

Proof. The 1st-order deformation cocycle $s \in H^1(X, \text{End}E)$ associated with the family E_t is nontrivial since the family E_t is nontrivial at $t = 0$. The nontrivial cocycle s is in the kernel of ρ^* since ρ^*s , the 1st-order deformation cocycle associated with the locally trivial family $\tilde{p} : \rho^*\mathcal{E} \rightarrow \Delta$, is trivial. \square

Lemma 3.24. *Let $p : X' \rightarrow X$ be an unramified Galois covering of a smooth projective manifold X and E a vector bundle on X . Then $H^0(\tilde{X}, \text{End}_0\rho^*E) \neq 0$ if one of the following holds:*

- 1) *The kernel of $p^* : H^1(X, \text{End}_0E) \rightarrow H^1(X', \text{End}_0\rho^*E)$ is nontrivial.*
- 2) *$H^0(X', \mathcal{O}) = \mathbb{C}$ and there is a pair of vector bundles E and F such that $p^*F = p^*E$ but $F \neq E \otimes \mathcal{O}(\chi)$ for any character $\chi : \pi_1(X) \rightarrow \mathbb{C}^*$.*

Proof. Assume that 1) holds then $H^0(\tilde{X}, \text{End}_0\rho^*E) \neq 0$ follows from lemma 3.12 (p must be an infinite unramified covering of X by Lemma 2.2).

If 2) holds then there is an isomorphism $i : p^*E \rightarrow p^*F$ and $F \neq E \otimes \mathcal{O}(\chi)$ for any character $\chi : G \rightarrow \mathbb{C}^*$. Let G be the Galois group of the covering. The isomorphism i is not G -equivariant since otherwise it would descent to an isomorphism $i' : E \rightarrow F$ on X . Consider the two possible cases: 1) there is a $g \in G$ such that $g^{-1}i^{-1}gi :$

$p^*E \rightarrow p^*E$ is a non-scalar endomorphism. Then $g^{-1}i^{-1}gi$ is a nontrivial element in $H^0(X', \text{End}_0 p^*E)$. 2) For all $g \in G$ the endomorphism $g^{-1}i^{-1}gi$ of p^*E is scalar. Since X' has no nonconstant holomorphic functions, the following holds: $g^{-1}i^{-1}gi = \chi(g)\text{Id}$, $\chi(g) \in \mathbb{C}^*$. Therefore, the map $\chi : G \rightarrow \mathbb{C}^*$ defines a character of G and $F = E \otimes \mathcal{O}(\chi)$ which can not happen, since it contradicts the assumption. \square

Let $\rho_0^* : \text{Mod}_0(X) \rightarrow \text{Vect}(\tilde{X})$ be the pullback map, where $\text{Mod}_0(X)$ is the moduli space of absolutely stable vector bundles on X . We denote points in $\text{Mod}_0(X)$ by the same letters as the corresponding vector bundles.

Proposition 3.25. *Let X be a projective manifold such that its universal cover \tilde{X} has no nonconstant holomorphic functions. If E is an absolutely stable vector bundle on X satisfying $H^0(\tilde{X}, \text{End}_0 \rho^* E) = 0$, then:*

- a) *The pullback map $\rho_0^* : \text{Mod}_0(X) \rightarrow \text{Vect}(\tilde{X})$ is a local embedding at E .*
- b) *For any absolutely stable bundle E there are only finite number of bundles F with $\rho^* E = \rho^* F$ and $E = F \otimes \mathcal{O}(\chi)$ with χ a character of $\pi_1(X)$.*

Proof. To prove part a) it is enough to show that the tangent map to ρ_0^* at E , $\rho^* : H^1(X, \text{End}E) \rightarrow H^1(\tilde{X}, \text{End} \rho^* E)$ is injective. The injectivity of $\rho^* : H^1(X, \text{End}E) \rightarrow H^1(\tilde{X}, \text{End} \rho^* E)$ follows from $H^0(\tilde{X}, \text{End}_0 \rho^* E) = 0$ and lemma 3.24 1).

Lemma 3.24 2) and the finiteness of the character group of $\pi_1(X)$ imply part b). The finiteness of the character group follows from $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. \square

To conclude, we consider the case when E is an absolute stable but $H^0(\tilde{X}, \text{End}_0 E) \neq 0$.

Theorem 3.26. *Let X be a projective manifold such that its universal cover \tilde{X} has no nonconstant holomorphic functions. If E is an absolute stable vector bundle on X satisfying $H^0(\tilde{X}, \text{End}_0 \rho^* E) \neq 0$ then associated to E is a normal subgroup $\pi_1(X') \subset \pi_1(X)$ corresponding to finite unramified covering $p : X' \rightarrow X$ with universal cover $\rho' : \tilde{X} \rightarrow X'$ satisfying:*

- i) $p^*E \simeq E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_m \otimes \mathcal{O}(\tau_m)$, $\tau_i : \pi_1(X') \rightarrow GL(k, \mathbb{C})$ with $k \geq 1$.
- ii) $H^0(\tilde{X}, \text{End}_0 \rho'^* E'_i) = 0$ for all $i = 1, \dots, m$.
- iii) *The natural action of the finite group $G = \pi_1(X)/\pi_1(X')$ on X' extends to the action on p^*E which permutes subbundles $E'_i \otimes \mathcal{O}(\tau_i)$ and this action gives the imbedding of G into S_m .*
- iv) *Let $G_1 \subset G$ be subgroup which acts identically on $E'_1 \otimes \mathcal{O}(\tau_1) \subset p^*E$ then $E'_1 \otimes \mathcal{O}(\tau_1)$ descends to the bundle $E'_1 \otimes \mathcal{O}(\tau'_1)$ on $X_1 = X'/G_1$ with $p_1 : X_1 \rightarrow X$ being a nonramified covering of degree $\text{rk}E / (\text{rk}E'_1 \otimes \mathcal{O}(\tau_1))$ and $E = p_* E'_1 \otimes \mathcal{O}(\tau'_1)$*

Proof. Consider the subsheaf \mathcal{A} of $\text{End} \rho^* E$ generated by the global sections of $\text{End} \rho^* E$. The sheaf $\text{End} \rho^* E$ is a sheaf of matrix algebras and \mathcal{A} is a sheaf of subalgebras since we can add and multiply sections. We claim that: $\mathcal{A} = H^0(\tilde{X}, \mathcal{A}) = H^0(\tilde{X}, \text{End} \rho^* E)$ is finite dimensional and isomorphic to a sum of m copies of the algebra of $k \times k$ matrices $M(k)$ for

a $k < r = \text{rk } E$; the action of $\pi_1(X)$ on the algebra A has no nontrivial $\pi_1(X)$ invariant ideals ($\pi_1(X)$ acts transitively on the m direct summands $M(k) \subset A = \bigoplus_{i=1}^m M(k)$). We also claim that the action of the algebra A on ρ^*E is such that each direct summand of A acts on each fiber $(\rho^*E)_x \simeq \mathbb{C}^r$, $x \in \tilde{X}$, as the same multiple $lM(k)$ of the standard representation of $M(k)$, $r = lmk$.

As mentioned above, the $\pi_1(X)$ -action on A permutes the m simple direct summands $M(k)$ and therefore gives a homomorphism $\sigma : \pi_1(X) \rightarrow S_m$. Associated with the normal subgroup $\text{Ker}(\sigma)$ is a finite unramified Galois covering of X , $p : X' \rightarrow X$, with $\pi_1(X') = \text{Ker}(\sigma)$. By construction the direct summands of A are $\pi_1(X')$ -invariant. Thus, $\rho^*E = \tilde{E}_1 \otimes \mathcal{O}^k \oplus \dots \oplus \tilde{E}_m \otimes \mathcal{O}^k$, where $\tilde{E}_i = \rho'^*E'_i$, E'_i is a vector bundle of rank l on X' and $\rho' : \tilde{X} \rightarrow X'$ is the universal cover of X' . On X' the bundle p^*E decomposes into $p^*E = E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_m \otimes \mathcal{O}(\tau_m)$ ($\tau_i : \pi_1(X') \rightarrow GL(k, \mathbb{C})$) giving i). Note that the group $G = \pi_1(X)/\pi_1(X')$ acts on p^*E permuting transitively the direct summands thus proving iii). Part ii) follows from $H^0(\tilde{X}, \text{End}(\rho'^*E'_i \otimes \mathcal{O}^k)) = M(k)$ since if $H^0(\tilde{X}, \text{End}_0 \rho'^*E'_i) \neq 0$ the group of global sections $H^0(\tilde{X}, \text{End}(\rho'^*E'_i \otimes \mathcal{O}^k))$ would be larger. Consider the group $G_1 \subset G$ which stabilizes $E'_1 \otimes \mathcal{O}(\tau_1)$. Then $E'_1 \otimes \mathcal{O}(\tau_1)$ descends to X_1/G_1 . The bundle p^*E also descends to X_1 and it decomposes into a direct sum $E'_1 \otimes \mathcal{O}(\tau_1) + E_N$. Let $p_1 : X_1 \rightarrow X$ be a corresponding covering of X . Consider the direct image $p_{1*}E'_1 \otimes \mathcal{O}(\tau_1)$ on X . We want to show that $p_{1*}E'_1 \otimes \mathcal{O}(\tau_1) = E$. Consider also $p_{1,*}p^*E$ which has a natural decomposition as $E + E_c$. Natural projection $i_1^* : p^*E \rightarrow E'_1 \otimes \mathcal{O}(\tau_1)$ which is identity on $E'_1 \otimes \mathcal{O}(\tau_1)$ induces a map $p_{1,*}p^*E \rightarrow p_{1,*}E'_1 \otimes \mathcal{O}(\tau_1)$. Denote by R the restriction of R on the direct summand $E \in p_{1,*}p^*E$. We want to show that R is an isomorphism. It follows from the fiberwise description of R . Let $C_x^{mlk} = \sum C_i^{mk}$ be a direct decomposition of the fiber of p^*E at $x \in X'$ into the sum of the fibers of the direct summands $E'_i \otimes \mathcal{O}(\tau_i)$ and $g_i \in G$ be the representatives of cosets G/G_1 . Then for a $x' \in X$ its pre-image $p_1^{-1}x' \subset X_1$ is equal to $\bigcup g_i x_1$ and the fiber of $E'_1 \otimes \mathcal{O}(\tau_1)$ over $g_i x_1$ is naturally isomorphic to C_i^{mk} . Now the map R becomes the trace map for the action of G on p^*E which implies that R is fiberwise isomorphism. This proves iv).

Claim: $A = H^0(\tilde{X}, \mathcal{A})$ is a subalgebra of the matrix algebra $M(r)$, $r = \text{rk}E$.

First, we prove the finite dimensionality of A . The sheaf \mathcal{A} is invariant under the action of $\pi_1(X)$ and defines a coherent subsheaf $\mathcal{A}' \subset \text{End}E$ on X with $\mathcal{A} = \rho^*\mathcal{A}'$. The absolute stability of E implies that $\det \mathcal{A}' \in -K_{eff}$ by corollary 3.8. The lemma 3.14 implies that \mathcal{A}' is isomorphic to the sheaf of sections of a flat vector bundle since $(\det \mathcal{A}')^{-k}$ has a nontrivial section for some k but $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. Hence $\mathcal{A} = \rho^*\mathcal{A}' = \mathcal{O}^q$. It follows from $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$, that the algebra A is finite dimensional.

We want to show that the algebra $A \cong \mathcal{A} \otimes k(x) \subset \text{End} \rho^*E \otimes k(x) \cong M(r)$, where x is any point in \tilde{X} and $k(x)$ is the residue field at x . Consider the exact sequence $0 \rightarrow \mathcal{A} \otimes \mathcal{I}(x) \rightarrow \mathcal{A} \rightarrow \mathcal{A} \otimes k(x) \rightarrow 0$, $\mathcal{I}(x)$ the ideal sheaf of the point x . Since \mathcal{A} is globally generated by its global sections it follows that the morphism $A = H^0(\tilde{X}, \mathcal{A}) \rightarrow H^0(X, \mathcal{A} \otimes k(x)) = \mathcal{A} \otimes k(x)$ is a surjection. If the morphism is also an injection we get

the desired isomorphism $A \cong \mathcal{A} \otimes k(x)$. The injectivity follows from $H^0(\tilde{X}, \mathcal{A} \otimes \mathcal{I}(x)) = 0$, which holds since the argument of the previous paragraph implies that any nontrivial section s of $\text{End} \rho^* E$ is nowhere vanishing.

Claim: The algebra A is semisimple.

The semisimplicity of A is equivalent to the maximal nilpotent ideal I_m of A being the zero ideal. The algebra A comes with a natural $\pi_1(X)$ -action. The maximal nilpotent ideal is a $\pi_1(X)$ -invariant ideal of A . Every nontrivial $\pi_1(X)$ -invariant ideal I of A defines naturally a nontrivial subsheaf $\mathcal{I}' \subset \mathcal{A}' \subset \text{End} E$. Suppose that the ideal I_m is nontrivial and consider the subsheaf $\mathcal{I}'_m E$ of E . The nilpotent condition, $I_m^k = 0$ for some k , implies that $\text{rk}(\mathcal{I}'_m E) < \text{rk} E$.

Consider the exact sequence $0 \rightarrow \mathcal{K} \rightarrow \mathcal{I}'_m \otimes E \rightarrow \mathcal{I}'_m E \rightarrow 0$. We will show that the subsheaf \mathcal{K} is an H -destabilizing subsheaf of $\mathcal{I}'_m \otimes E$ for any polarization of X . This can not be, since E and \mathcal{I}'_m are H -semistable (\mathcal{I}'_m is a flat bundle) and hence $\mathcal{I}'_m \otimes E$ is also H -semistable (the tensor product of two semistable sheaves is semistable). We need to get the destabilizing inequality $[\text{rk}(\mathcal{I}'_m \otimes E) \det \mathcal{K} - \text{rk} \mathcal{K} \det(\mathcal{I}'_m \otimes E)] \cdot H^{n-1} > 0$. Using $\det \mathcal{K} = \text{rk} \mathcal{I}'_m \det E - \det \mathcal{I}'_m E$ and $\text{rk} E \det \mathcal{I}'_m E - \text{rk} \mathcal{I}'_m E \det E \in K_{eff+}$ (E is an absolute stable bundle), it follows that $\text{rk}(\mathcal{I}'_m \otimes E) \det \mathcal{K} - \text{rk} \mathcal{K} \det(\mathcal{I}'_m \otimes E) \in -K_{eff+}$. Hence we obtain the desired contradiction, which implies that the $\pi_1(X)$ -invariant ideal I_m must be the zero ideal and A is semisimple.

Claim: A has no proper $\pi_1(X)$ -invariant ideals.

We proved that A is semisimple and hence $A = \sum_{i=1}^m M(r_i)$ with $r_1 + \dots + r_m \leq r$. First, we note that if I is an ideal of $A = \sum_{i=1}^l M(n_i)$ (A acts on \mathbb{C}^r) such that $IC^r = \mathbb{C}^r$ then $I = A$. We prove the claim by showing that for any nonzero $\pi_1(X)$ -invariant ideal I of A the equality $I = A$ or equivalently $\mathcal{I}' E = E$ must hold. If $\text{rk} \mathcal{I}' E = \text{rk} E$ then at some $x \in \tilde{X}$, $I_x = I \otimes k(x) \subset \text{End}(\rho^* E)_x$ is such that $I_x(\rho^* E)_x = (\rho^* E)_x$ and hence $I = A$. If I is such that $\text{rk} \mathcal{I}' E < \text{rk} E$ then the argument of the paragraph above proves that $I = 0$ and hence also the claim.

Claim: The algebra A is equal to $mM(k)$ and the representation of each $M(k)$ is a multiple of a standard rank k representation of $M(k)$.

The algebra A is as noted before $A = \sum_{i=1}^m M(r_i)$ with $r_1 + \dots + r_m \leq r$. Since each $M(r_i)$ is simple it follows that the action of $\pi_1(X)$ preserves the ideals of A corresponding to the sums of all the $M(r_i)$ with r_i equal to a fixed k . Therefore all the r_i are equal to the same k since any $\pi_1(X)$ -invariant ideal I of A is either trivial or the full A .

Finally, we show that the representation of each $A = M(k)$ in $M(n)$ is a multiple of the standard representation. Any irreducible representation of $M(k)$ is the standard representation or the zero representation. The presence of a zero representation as an irreducible component of the representation of $M(k)$ in $M(n)$ would imply that $\mathcal{A}' E \neq E$ which is not possible from the discussion above.

□

The following lemma that follows from our results and observation 3.20.

Lemma 3.27. *If \tilde{X} has no nonconstant holomorphic functions then $H^1(X, \mathbb{C}(\tau)) = H^1(X, \mathcal{O}(\tau)) = 0$ for any linear representation τ of $\pi_1(X)$.*

Proof. Let $f : X' \rightarrow X$ of X be the covering corresponding to the kernel $G \subset \pi_1(X)$ of the representation τ . The hypothesis $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ and observation 3.20 imply that τ is finite. It also follows from $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ that $H^1(X', \mathbb{C}^k) = H^1(X', \mathcal{O}^k) = 0$. Since τ is finite the covering f is also finite and hence the conclusion follows from the imbeddings $f^* : H^1(X, \mathbb{C}(\tau)) \rightarrow H^1(X', \mathbb{C}^k)$ and $f^* : H^1(X, \mathcal{O}(\tau)) \rightarrow H^1(X', \mathcal{O}^k)$. □

Theorem B. *Let X be a projective manifold whose universal cover has only constant holomorphic functions. Then:*

- a) *The pullback map $\rho_0^* : \text{Mod}_0(X) \rightarrow \text{Vect}(\tilde{X})$ is a local embedding.*
- b) *For any absolutely stable bundle E there are only finite number of bundles F with $\rho^*E = \rho^*F$.*
- c) *Moreover, there is a finite unramified cover $p : X' \rightarrow X$ associated with E of degree $d \leq \text{rk}E!$ with universal covering $\rho' : \tilde{X} \rightarrow X'$. On X' there is a collection of vector bundles $\{E'_i\}_{i=1, \dots, m}$ on X' with $H^0(\tilde{X}, \text{End}_0 \rho'^* E'_i) = 0$ such that $\rho^*F \simeq \rho^*E$ if and only if:*

$$p^*F = E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_m \otimes \mathcal{O}(\tau_m)$$

The bundles $\mathcal{O}(\tau_i)$ are flat bundles associated with finite linear representations of $\pi_1(X')$ of a fixed rank k with $\text{rk}E|k$.

Proof. The case for vector bundles E such that $H^0(\tilde{X}, \text{End}_0 \rho^*E) = 0$ was done in proposition 3.25. We proceed to consider the case $H^0(\tilde{X}, \text{End}_0 \rho^*E) \neq 0$.

a) Let $\text{Mod}_0(X, V)$ be the moduli space of absolutely stable bundles with the same Chern classes as V . The formal tangent space of $\text{Mod}_0(X, V)$ at V is given by $H^1(X, \text{End}V)$. The vector bundle $\text{End}V$ is semistable with $\det \text{End}V = \mathcal{O}$ and is the direct sum $\text{End}E = \bigoplus_{i=1}^l F_i$ of absolutely stable bundles with $\mu_H(F_i) = 0$ by corollary 3.10. The kernel of the tangent map $\rho^* : H^1(X, \text{End}V) \rightarrow H^1(\tilde{X}, \text{End} \rho^*V)$ is the direct sum of the kernels of $\rho_i^* \equiv \rho^* : H^1(X, F_i) \rightarrow H^1(\tilde{X}, \rho^*F_i)$. Proposition 3.16 implies that if $\ker \rho_i^* \neq 0$ then $F_i = \mathcal{O}(\tau)$. Hence the kernel of the tangent map ρ^* is trivial since it follows from lemma 3.27 that $H^1(X, F_i) = 0$. This implies that ρ^* is a local imbedding.

Part b) is a consequence of c), hence we first consider c). Theorem 3.26 states ρ^*E has the decomposition $\rho^*E \cong \tilde{E}_1 \otimes \mathcal{O}^k \oplus \dots \oplus \tilde{E}_m \otimes \mathcal{O}^k$ with simple vector bundles \tilde{E}_i . If $\rho^*F \cong \rho^*E$ then ρ^*F inherits also a decomposition $\rho^*F \cong \tilde{F}_1 \otimes \mathcal{O}^k \oplus \dots \oplus \tilde{F}_m \otimes \mathcal{O}^k$ with $\tilde{F}_i \otimes \mathcal{O}^k \cong \tilde{E}_i \otimes \mathcal{O}^k$. Since the \tilde{E}_i are simple vector bundles on \tilde{X} it follows that $\tilde{F}_i \cong \tilde{E}_i$. Also by theorem 3.26, we have a finite covering $p : X' \rightarrow X$ where p^*E decomposes as described in the theorem and equally p^*F decomposes into $p^*F \cong F'_1 \otimes \mathcal{O}(\tau'_1) \oplus \dots \oplus$

$F'_m \otimes \mathcal{O}(\tau'_m)$ with $\rho'^* F'_i = \tilde{F}_i$. It follows from lemma 3.24 that $p^* E'_i \otimes \mathcal{O}(\chi) \cong p^* F'_i$ for some character $\chi : \pi_1(X') \rightarrow \mathbb{C}^*$, since $\rho'^* E'_i = \tilde{E}_i \cong \tilde{F}_i = \rho'^* F'_i$ and \tilde{E}_i is simple. Hence c) follows from the decomposition for $p^* E$.

To prove b) we first claim that there is a finite unramified Galois covering $\hat{p} : \hat{X} \rightarrow X$ associated with E such that $\rho^* F \cong \rho^* E$ if and only if $\hat{p}^* F \cong \hat{p}^* E$. The previous paragraph states that if $\rho^* F \cong \rho^* E$ then:

$$p^* F \cong E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_m \otimes \mathcal{O}(\tau_m) \quad (3.3)$$

where $\mathcal{O}(\tau_i)$ are flat bundles of rank k associated with finite representations (not the same as in the theorem 3.26). The variety of representations of $\pi_1(X') \rightarrow GL(k, \mathbb{C})$ for a fixed k , $M(\pi_1(X'), k)$, is a finite set of points, since by lemma 3.27 $H^1(X', \text{End}\mathcal{O}(\tau)) = 0$ for all representations τ and hence $M(\pi_1(X'), k)$ is zero dimensional. The finiteness of the set of representations implies the existence of a finite Galois cover $\hat{p} : \hat{X} \rightarrow X$ where $\hat{p}^* F \cong \hat{p}^* E$ if $\rho^* F \cong \rho^* E$. The result follows then by the lemma:

Lemma 3.28. *Let $g : Y \rightarrow X$ be a finite unramified Galois covering of X and E an absolutely stable bundle on X . If F is a vector bundle on X such that $g^* F \cong g^* E$ then F belongs to a finite collection of isomorphism classes of vector bundles on X .*

Proof. If $g^* E$ is a simple vector bundle then the proof of lemma 3.24 gives the result. More precisely, it shows that $F \cong E \otimes \mathcal{O}(\chi)$ where $\chi : G \rightarrow \mathbb{C}^*$ is a character of the Galois group G of the cover f .

If $f^* E$ is not simple applying the argument in theorem 3.26 we get that $g^* E \cong E_1 \otimes \mathcal{O}^k \oplus \dots \oplus E_m \mathcal{O}_k$ and $H^0(Y, \text{End}g^* E) = \bigoplus_{i=1}^m M(k)$ for some k dividing $\text{rk}E$. The vector bundles E and F are quotients of two different actions of the Galois group G on $p^* G$. The quotient of action of G on $p^* E$ is up to isomorphism determined by the isomorphism class of induced representation $\tau : G \rightarrow GL(mk, \mathbb{C})$. Our result follows since the number of isomorphism classes of representations $\tau : G \rightarrow GL(mk, \mathbb{C})$ is finite.

□

□

Remark: We have a similar result for H -stable bundles if \tilde{K}_{eff} satisfies $P1$ or $P1'$.

What about the map of the space of all bundles (omitting the discussion of whether it can be well defined)? Notice that for any given filtration of saturated subsheaves in a vector bundle V there is a blow up X' of X such that the pullback of this filtration becomes a filtration of vector bundles (see Moishezon [Mo69] lemma 3.5). In particular, for any vector bundle V on X one can use the Harder-Narasimhan filtration. Since the algebra of holomorphic functions on \tilde{X} does not change after changing blowing up, any conclusion about the function theory for \tilde{X}' holds for \tilde{X} . It follows from the above that if $P1'$ holds then the pullback map for all bundles is non-injective modulo representations of $\pi_1(X)$ only if there are cocycles $\alpha \in H^1(X, V)$ such that $\rho^* \alpha = 0$.

The following are some remarks about how to use the above results to show that the universal cover of a projective variety has a nonconstant holomorphic function.

Proposition 3.29. *Let X be a projective manifold of dimension n and X' be an infinite unramified cover of X then $H^n(X', \mathcal{F}) = 0$ for any coherent sheaf \mathcal{F} on X' .*

Proof. The result follows from Čech cohomology and Leray coverings if any noncompact cover of a n -dimensional projective variety is covered by n Stein open subsets. Pick $n - 1$ generic hyperplane sections H_i and let $C = H_1 \cap \dots \cap H_{n-1}$. By Lefschetz theorem C is a smooth curve such that $\pi_1(C) \rightarrow \pi_1(X)$ is a surjection. This implies that the pre-image of C in X' is an irreducible noncompact curve C' . Hence C' is Stein (Behnke-Stein theorem). The infinite cover X' is covered by the pre-images U_i of $X \setminus H_i$ in X' and a neighborhood of C' . The pre-images U_i are Stein open subsets of X' since any unramified cover of a Stein manifold is Stein. To conclude, C' has an open Stein neighborhood in X' since C' is a Stein closed subvariety of X' (Siu [Si76]). \square

Remark: Proposition 4.26 implies that for surfaces the structure of the space of the moduli space of vector bundles on \tilde{X} should be similar to the structure of the moduli space of vector bundles on a curve. Namely the groups $H^2(\tilde{X}, \mathcal{F})$ vanish for any coherent sheaf \mathcal{F} . In particular, there are no algebraic obstructions in $H^2(\tilde{X}, \text{End}E)$ to deform a vector bundle E along a cocycle in $H^1(\tilde{X}, \text{End}E)$ though there may be an analytic one (problem of convergency). We expect that any bundle of rank ≥ 2 has a complete flag of subbundles if there is a complete flag of topological subbundles. This would imply that the K-group $K_0(\tilde{X})$ reduces to $\text{Pic}(X) \times \mathbb{Z}$. The above motivates the authors' expectation that many different bundles on X coincide after pulling back to \tilde{X} . This remark may also provides a clue to the proof of $H^0(\tilde{X}, \mathcal{O}) \neq \mathbb{C}$ for the universal covers \tilde{X} of projective surfaces X with infinite fundamental group. We plan to address this in the next publication.

4. GEOMETRIC VANISHING THEOREM FOR NEGATIVE BUNDLES

The arguments used in the proof of Theorem A can be used to give an alternative proof of the vanishing theorem for negative vector bundle V over a projective manifold X whose $\text{rk } V < \dim X$.

Theorem 4.1. *If V is a negative vector bundle on a complex manifold X with $\text{rk } V < \dim X$, then $H^1(X, V) = 0$.*

Proof. Suppose it exists a nontrivial $s \in H^1(X, V)$ and let:

$$0 \rightarrow V \rightarrow V_s \rightarrow \mathcal{O} \rightarrow 0$$

be the associated extension. As in the Theorem A, consider the dual exact sequence and $A_s = \mathbb{P}(V_s^*) \setminus \mathbb{P}(V^*)$ be an affine bundle, which by the negativity of V is strictly

pseudoconvex. Let $r : A_s \rightarrow St(A_s)$ be the Remmert reduction, where r is proper contracting $M = \cup_{i=1}^k M_i$ and $St(A_s)$ is a Stein space with isolated singularities.

The aim is to obtain a contradiction from topological conditions. The Stein space $St(A_s)$ has $\dim_{\mathbb{C}} St(A_s) = \dim_{\mathbb{C}} X + r$ and hence it has the homotopy type of a simplicial complex of real dimension at most equal to $\dim_{\mathbb{C}} X + r$. On the other hand, $St(A_s) = A_s / (\coprod_i M_i)$ as a topological space and so for the reduced singular homology of A_s $\tilde{H}_i(St(A_s), \mathbb{C}) = H_i((A_s, \coprod_i M_i), \mathbb{C})$. Now the long exact homology sequence of the pair $(A_s, \coprod_i M_i)$ together with the fact that $\coprod_i M_i$ is compact of complex dimension strictly less than $\dim_{\mathbb{C}} X = n$ (by proposition 2.5) gives that $H_{2n}(A_s, \mathbb{C}) \cong \tilde{H}_{2n}(St(A_s), \mathbb{C}) = H_{2n}(St(A_s), \mathbb{C})$.

In conclusion, $St(A_s)$ as a Stein manifold of $\dim_{\mathbb{C}} St(A_s) = n + r < 2n$ must have $H_{2n}(St(A_s), \mathbb{C}) = 0$. The previous argument gives $H_{2n}(St(A_s), \mathbb{C}) \cong H_{2n}(A_s, \mathbb{C})$. The contradiction follows since A_s as an affine bundle over X is homotopically equivalent to X and therefore $H_{2n}(A_s, \mathbb{C}) \cong H_{2n}(X, \mathbb{C}) \neq 0$.

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