Complex cobordism and embeddability of CR-manifolds

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This paper studies (directed) complex cobordisms between compact three dimensional strictly pseudoconvex Cauchy-Riemann manifolds. Assume the complex cobordism between the compact strictly pseudoconvex Cauchy-Riemann manifolds M_1 and M_0 is given by a compact complex manifold X with boundary having M_1 as its pseudoconvex end and M_0 as its pseudoconcave end. We answer the following questions: if one of the ends is an embeddable CR-manifold is the other end also embeddable? Are all CR-functions on the pseudoconvex end of X boundary values of holomorphic functions on the interior of X? If the complex cobordism manifold X is of dimension greater than 2, the affirmative answer to these questions follows immediately from the embeddability of all strictly pseudoconvex CR-manifolds of dimension greater than 3. For complex cobordisms of dimension 2 these questions were still open. The ends of complex cobordism 2-manifolds are strictly pseudoconvex CR-manifolds of dimension 3 whose embeddability is no longer guaranteed and is a topic of active research.

To answer the questions raised above, we produce two new methods to construct pseudoconcave surfaces, whose boundaries are non-embeddable strictly pseudoconvex CR-manifolds of dimension 3. The constructions allow us to show that:

1) There are complex cobordant 3-dimensional strictly pseudoconvex CR-manifolds M_1 and M_0, with one embeddable and the other not.

2) There are complex cobordism 2-manifolds X with CR-functions on the pseudoconvex end M_1 that are not boundary values of holomorphic functions on the interior of X.

3) There are complex cobordism 2-manifolds X with embeddable pseudoconvex ends M_1, where the extendabity of all the CR-functions on M_1 to holomorphic functions on X does not imply the embeddability of the pseudoconcave end M_0.

4) There are holomorphic families of complex cobordism 2-manifolds where all the members have their pseudoconvex ends embeddable but only the central member has the pseudoconcave end embeddable.

The first result answers negatively the main question of this paper on the preservation of embeddability under complex cobordisms. In the literature this question appeared in

[EpHe1-00]. Previous to our result, the only answer was positive [EpHe01], but the complex cobordism manifold X had an extra condition. It was required that X had a strictly plurisubharmonic function φ and that its ends M_1 and M_0 were level sets of φ.

The second result gives the first case where CR-functions on the pseudoconvex end M_1 of a complex manifold X, whose ends are strictly pseudoconvex CR-manifolds, do not extend to holomorphic functions on X. The basic results on extendability are in the other direction: the local extendability result of Lewy states that all CR-functions of the pseudoconvex end M_1 extend to an holomorphic function on a small neighborhood of M_1 and the global extendability result of Kohn and Rossi [KoRo65] implying that if all the other ends of X are embeddable then all the CR-functions on M_1 extend. This last global result shows in particular that result 2) is only possible in dimension 2.

The third result clarifies the strength of the connection between the embeddability of the pseudoconcave end M_0 of a complex cobordism manifold X having an embeddable pseudoconvex end M_1 and the extendability of all the CR-functions of M_1 to holomorphic functions on X. The extendability of the CR-functions from the pseudoconvex end M_1 is necessary for embeddability of M_0, as follows from [KoRo65], but the result states that it is not sufficient. The fourth result states that the property that a complex cobordism manifold preserves embeddability is not stable under small deformations. The last two results also say that complex cobordism manifolds not preserving embeddability can be quite similar to complex cobordism manifolds that do.

A compact $(2n+1)$-dimensional $Cauchy-Riemann\ manifold$ (CR-$(2n+1)$-manifold) consists of: a compact $(2n+1)$-dimensional manifold M, a rank n complex subbundle $T^{0,1}M \subset TM \otimes \mathbb{C}$ satisfying $T^{1,0}M \cap T^{0,1}M = \{0\}$ ($T^{1,0}M \equiv \overline{T^{0,1}M}$) and the integrability condition $[\overline{Z}, \overline{Z'}]|_p \in T_p^{0,1}M$ for local sections $\overline{Z}, \overline{Z'}$ of $T^{0,1}M$. If the CR-$(2n+1)$-manifold has the additional property that any nonvanishing local section $\overline{Z}$ of $T^{0,1}M$ is such that $[Z, \overline{Z}]|_p \notin T_p^{0,1}M \oplus T_p^{1,0}M$ for any $p \in M$, it is called $strictly\ pseudoconvex$ (SPCR-$(2n+1)$-manifold).

A differentiable function $f : M \to \mathbb{C}$ is said to be a $CR-function$ if it is in the kernel of the $\overline{\partial}_b$ operator, $\overline{\partial}_b f = df|_{T^{0,1}M}$. That is, it verifies the tangential Cauchy-Riemann equations $\overline{Z}f = 0$ for all local sections of $T^{0,1}M$.

Definition A CR-$(2n+1)$-manifold M is embeddable if there is a collection of CR functions $\{f_1, ..., f_N\}$ giving an embedding $f = (f_1, ..., f_N) : M \to \mathbb{C}^N$.

The origin of CR-structures lies in the study of real hypersurfaces in $\mathbb{C}^n$. The CR-structure being given by $T^{0,1}M = T^{0,1}\mathbb{C}^n \cap TM \otimes \mathbb{C}$. A CR-function is a function that satisfies the differential conditions of holomorphy that can be verified along M. An important example of SPCR-manifolds are the boundaries of strictly pseudoconvex domains in $\mathbb{C}^n$.

Definition A CR-$(2n+1)$-manifold M is fillable if it is the boundary of a normal complex Stein space X with the CR-structure induced from the complex structure of X. A filling of M is any normal complex space Y which is a modification of a Stein space and whose

boundary is M.

An example of Andreotti and Grauert, see [Ro65] showed that not all compact SPCR-3-manifolds are embeddable. Later Boutet de Monvel [Bou74] showed that for $n \geq 2$ any compact SPCR-(2n+1)-manifold M is embeddable, this follows from the result of Kohn stating that the range of $\overline{\partial}_b$ is closed plus the result of [Bou74] stating that if the range of $\overline{\partial}_b$ is closed then M is embeddable. The following important result also holds: fillability of any compact SPCR-(2n+1)-manifold M is equivalent to the embeddability of M. This result follows from [Bou74], [Ko86] and also [HaLa75]. A significant amount of work has been produced to understand the structure of the set of embeddable SPCR-structures for 3-manifolds by Bland, Burns, Catlin, Epstein, Henkin and Lempert.

Acknowleddments I would like to thank Professor Charles Epstein for generously sharing his knowledge with me and for his enthusiasm and Professor Yum-Tong Siu for always inspiring and encouraging me and the hospitality of the staff at Harvard University.

SECTION 1

The CR-structure on a smooth real hypersurface M given as a level set of a strictly plurisubharmonic function φ on a complex manifold V, $M = \{\varphi^{-1}(c)\}$ and $d\varphi(p) \neq 0$ on $T_p M$ for all $p \in M$, is strictly pseudoconvex. In the opposite direction, let M be a compact smooth boundary component of complex manifold with boundary X such that the induced CR-structure on M is strictly pseudoconvex then M is a nondegenerate level set of a strictly plurisubharmonic function φ defined in a neighborhood of M, $V \subset X$. M is said to be a *pseudoconvex (pseudoconcave) end* of X if $M = \{\varphi^{-1}(0)\}$ and $X \cap V = \{\varphi(x) \leq 0\}$ ($X \cap V = \{\varphi(x) \geq 0\}$).

Definition Let X be a connected compact complex manifold with a smooth boundary. If $\partial X = M_1 \amalg M_0$ with M_0 a disjoint union of pseudoconcave ends and M_1 a disjoint union of pseudoconvex ends, then M_0 is said to be complex cobordant to M_1 and X is a (directed) complex cobordism manifold.

A special case of (directed) complex cobordisms was considered in [EpHe01]. Two CR-manifolds M_1 and M_0 are called *strictly* CR-*cobordant*, if there is complex cobordism manifold X with $\partial X = M_1 \amalg M_0$ having a strictly plurisubharmonic function, $\phi : X \to \mathbb{R}$, with M_1 and M_0 as level sets $\phi^{-1}(1)$ and $\phi^{-1}(0)$ respectively. X is called a *strict complex cobordism manifold*. From now on, all complex cobordisms will be directed complex cobordisms.

A compact complex manifold with a smooth boundary is called a *pseudoconvex* (pseudo-concave) manifold if all its boundary components are pseudoconvex (pseudoconcave) ends. In [KoRo65] it is shown that a pseudoconvex manifold has only one end. This implies that a complex cobordism manifold X of dimension greater than 2 can not have more than one pseudoconvex end. The complex manifold X' obtained from X by filling its pseudoconcave

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ends is pseudoconvex. In Corollary 1, we show that for dimension 2 it is possible to have complex cobordisms manifolds with more than one pseudoconvex ends.

On the other end it is possible for a pseudoconcave manifold Y to have more than one end. Note that if Y is a pseudoconcave manifold and M and M' are SPCR-manifolds such that $\partial Y = M \amalg M'$, the above definition of directed cobordism does not apply and M is not considered to be complex cobordant to M' or vice-versa.

The main goal of this paper is to understand how the embeddability of one end of a complex cobordism manifold X influences the embeddability of the other end. The first observation is that if the pseudoconcave end M_0 of X is embeddable then the pseudoconvex end M_1 is also embeddable. The embeddability of M_1 follows from its fillability, the manifold $S' = X \amalg_{M_0} S$, S a filling of M_0, is a filling of M_1. The other direction is the interesting case.

Prior to our work, the existing result on whether the embeddability of the pseudoconcave end implies the embeddability of the pseudoconvex end was in the positive direction. The result was for the special case of strictry CR-cobordant SPCR-manifolds [EpHe01]. In [EpHe01] it was shown that if the SPCR-manifolds M_1 and M_0 are strictly CR-cobordant then M_1 is embeddable iff M_0 is embeddable. One of the main ingredients of the proof is that by using the Lewy extension result all the CR-functions on M_1 are boundary values of holomorphic functions on X. In the next two sections we show that for a general complex cobordism M_1 being embeddable does not imply that M_0 is embeddable, and the same holds even if all CR-functions on M_1 are boundary values of holomorphic functions on the complex cobordism manifold.

It was mentioned in the introduction that the fillability is intimately related to the embeddability for SPCR-manifolds. It is therefore fruitful to relate a complex cobordism manifold with the fillings of its pseudoconvex end.

Definition Let M_1 be an embeddable SPCR-manifold. If M_1 is the pseudoconvex end of a complex cobordism manifold X whose other ends are pseudoconcave, then X is called a partial filling of M_1. We say that we have rigidity on the partial fillings of M_1 if they are all modifications of open subsets of the unique Stein normal space filling M_1.

Let M_1 be an embeddable SPCR-manifold of dimension greater than 3 then all partial fillings extend to an actual filling of M_1 and they are rigid. Let X be a complex cobordism manifold with the pseudoconvex end M_1 and the pseudoconcave ends $M_1 \amalg ... \amalg M_k$. The embeddability of all SPCR-manifolds of dimension greater than 3 implies that the M_i, $i = 1, ..., k$, are embeddable or equivalently fillable, then X is an open subset of a filling S' of M_1, $S' = X \amalg_{(M_0 \amalg ... \amalg M_k)} (S_1 \amalg ... \amalg S_k)$ with S_i a filling of M_i. In [Gr62] it is shown that all fillings of M_1 are proper modifications of its unique Stein normal filling.

The question of embeddability being preserved under complex cobordisms can be translated to: can we start filling an embeddable SPRC-3-manifold M_1 in the wrong way (not extendable to a filling)? A distinct but also relevant question is: what is the freedom in the partial fillings of M_1? Theorem 1 of the next section answers the first question affirmatively and from Theorem 3 of section 3 it follows that there can be plenty of freedom in the partial

fillings of M_1.

Remark: It is interesting to re-examine the result of Epstein and Henkin [EpHe01] described above. The result implies that, regardless of the dimension, in any partial filling of M_1 there is a sufficiently small collar of M_1 in X that is rigid.

SECTION 2

In this section we give a method to construct complex cobordisms where one end is embeddable but the other is not. We also show that the complex cobordisms produced by this method are very different, in terms of function theory, from complex cobordisms where all ends are embeddable.

The next lemma will be used to construct pseudoconcave surfaces containing disjoint positive curves.

Lemma 1 Let X be a nonsingular complex surface containing the normal crossing divisor $C = C_1 \cup ... \cup C_k$, with C_i nonsingular positive compact curves, i.e. $C_i^2 > 0$. Then there exist a neighborhood W of C where the line bundle $\mathcal{O}(C)$ associated to the divisor C is positive and a smooth strictly plurisubharmonic function $\varphi : W \setminus C \to \mathbb{R}$ such that:

i) For every sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \in W \setminus C$ converging to $x \in C$, $\lim_{x_n \to x} \varphi(x_n) = +\infty$.

ii) For $c \gg 0$, $X_c = \{x \in W : \varphi(x) \geq c \text{ or } x \in C\}$ is a relatively compact pseudoconcave neighborhood of C.

Proof: First we show that if the line bundle $\mathcal{O}(C)$ is positive in a neighborhhod W of C then we can construct a strictly plurisubharmonic function $\varphi : W \setminus C \to \mathbb{R}$ satisfying i) and ii). A line bundle L over X is positive at $x \in X$ if L has a hermitean metric h such that the associated real $(1,1)$-form $\Omega_h = \sqrt{-1}\Theta_h = -\sqrt{-1}\partial\overline{\partial} \log h$ is positive at x (Θ_h is the curvature form for h).

The line bundle $\mathcal{O}(C)$ is defined by the transition functions $g_{ij} = \frac{f_j}{f_i}$, where $\{f_i \in O(U_i)\}$ are the defining equations of C for the covering $\{U_i\}$ of X. A hermitian metric h on $\mathcal{O}(C)$ is given by a collection of positive functions, $h_i : U_i \to \mathbb{R}$ satisfying $h_j = h_i \frac{|f_i|^2}{|f_j|^2}$ on $U_i \cap U_j$. If the hermitian metric h on $O(C)$ is such that the form Ω_h positive at all $x \in W$, then the function $\varphi : W \setminus C \to \mathbb{R}$ defined locally by:

$$\varphi|_{U_i \setminus \{f_i = 0\}} \equiv \varphi_i = -\log(h_i |f_i|^2)$$

is strictly plurisubharmonic and satisfies the properties i) and ii). The positivity of the form Ω_h on W implies that φ is strictly plurisubharmonic on $W \setminus C$. Since $\partial\overline{\partial}\varphi_i = \partial\overline{\partial}(-\log(h_i))$, where $f_i \neq 0$, the positive definiteness of the Levi form of φ_i follows from the positivity of the $(1,1)$-form Ω_h associated with the metric h. The properties i) and ii) follow directly from the definition of φ plus the compactness of the C_i's.

We proceed to show that $\mathcal{O}(C)$ is positive in a neighborhood of C. The first step is to construct a metric h on $\mathcal{O}(C)$ such that $\Omega_{h|_{C_i}} > 0$ on each irreducible component C_i. The map $i_{\mathcal{O}_C(mC)} : C \to \mathbb{P}^N$ defined by a basis of $H^0(C, \mathcal{O}_C(mC))$ is an embedding for each irreducible component C_i if $m \gg 0$, since $C_i^2 > 0$ ($\mathcal{O}_C(mC)$ is the restriction to C of the line bundle associated to the divisor mC on X). If $\tilde{h}$ is a metric on $\mathcal{O}_{\mathbb{P}^N}(1)$ with positive $(1,1)$-form $\Omega_{\tilde{h}}$ then the m-square root of the pullback $h' = i^*_{\mathcal{O}_C(mC)} \tilde{h}$ is a metric for $\mathcal{O}_C(C)$ satisfying $\Omega_{h'|_{C_i}} > 0$ on each C_i. Let h be a metric of $\mathcal{O}(C)$ on X that is a C^∞ extension of h'. The following steps provide us with a method to modify the metric h and obtain a metric h'' with the desired curvature properties on a neighborhood of C.

Let $F = \{x_k\}_{k=1,\ldots,l}$ be the collection of the normal crossing points of C. Choose an open covering $\{U_i\}$ of X consisting of relatively compact open subsets of X with local coordinates z_1^i, z_2^i such that the defining equations f_i are of the form: 1) $f_i = 1$, if $C \cap U_i = \emptyset$; 2) $f_i = z_1^i$, if $C \cap U_i \neq \emptyset$ and no $x_k \in U_i$; 3) $f_i = z_1^i z_2^i$, if $C \cap U_i \neq \emptyset$ and some $x_k \in U_i$.

We change h to a new metric h' such that $\Omega_{h'}|_{T_{x_k}X}$ is a positive $(1,1)$-form at the crossing points x_k. If $x_k \in U_i$ and the ball $B_{x_k}(r)$ is such that $B_{x_k}(r) \cap U_j = \emptyset$ for $j \neq i$. We change h_i to:

$$h_i' = h_i e^{2Re(az_1 \bar{z}_2) \cdot \rho}$$

where $a = -\frac{\partial^2 \log h_i(x_k)}{\partial \bar{z}_2 \partial z_1}$ and ρ is a smooth function on U_i with value 1 on $B_{x_k}(\frac{r}{2})$ and value 0 outside $B_{x_k}(r)$. The functions h_i' and h_j, $j \neq i$ give a metric, h', on $O(C)$, since $h_i'|_{U_i \cap U_j} = h_i|_{U_i \cap U_j}$ for all $j \neq i$. The new metric h' is equal to h on C, $h_i'|_{C \cap U_i} = h_i|_{C \cap U_i}$, and has the associated $(1,1)$-form:

$$\Omega_{h'}(x_k) = -\sqrt{-1}\partial_1 \bar{\partial}_1 \log h_i(x_k) - \sqrt{-1}\partial_2 \bar{\partial}_2 \log h_i(x_k),$$

since the modification eliminated the crossed terms. The positivity of $\Omega_{h|_{C_i}}$ on each C_i implies that $-\frac{\partial^2 \log h_i(x_k)}{\partial z_1 \partial \bar{z}_1}, -\frac{\partial^2 \log h_i(x_k)}{\partial z_2 \partial \bar{z}_2} > 0$ and therefore $\Omega_{h'}|_{T_{x_k}X}$ is a positive $(1,1)$-form.

To finish, we change h' to a metric h'' which has the curvature form $\Omega_{h''}$ positive near the curve C. The modification was already done for the smooth case in [Sc73] and consists of changing the h_i to:

$$h_i'' = \frac{h_i'}{1 + ch_i'|f_i|^2}$$

This change is well defined since the collection functions $1 + ch_i'|f_i|^2$ on the U_i define a global positive smooth function on X. The associated $(1,1)$-form $\Omega_{h''}$ changes accordingly to:

$$\Omega_{h_i''} = \Omega_{h_i'} - \sqrt{-1}\partial\bar{\partial}\log(1 + ch_i'|f_i|^2)$$

The contribution of $\partial\bar{\partial}\log(1 + ch_i'|f_i|^2)$ on $C \cap U_i$ is: 1) $ch_i dz_1^i \wedge d\bar{z}_1^i$ if $f_i = z_1^i$; 2) $ch_i|z_2^i|^2 dz_1^i \wedge d\bar{z}_1^i + ch_i|z_1^i|^2 dz_2^i \wedge d\bar{z}_2^i$ if $f_i = z_1^i z_2^i$.

We claim that for $c \gg 0$ this new metric has the desired properties. It is enough to check that for $c \gg 0$ the form $\Omega_{h'}|_{T_xX}$ is positive, i.e. the matrix $[\omega_{jk}](x) = [-\frac{\partial^2 \log h_i'}{\partial z_j^i \partial \bar{z}_k^i}(x)]$ is positive definite, for all $x \in C$.

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In case 1) ($f_i = z_1^i$) for $x \in C \cap U_i$ the off diagonal entries of $[\omega_{jk}](x)$ and the second diagonal entry do not depend on c. On the other hand, the first diagonal entry can be made as big as desired by increasing c making the matrix $[\omega_{jk}](x)$ positive definite, since the second diagonal entry is positive ($\Omega_{h|_{C_i}}$ is positive). In case 2) ($f_i = z_1^i z_2^i$) the same argument works for the points that are not the crossing point . The positivity of the matrix $[\omega_{jk}]$ for all $x \in C$ and $c \gg 0$ follows from the compactness of C and the previously established positivity of $\Omega_{h'}|_{T_{x_k}X}$ at the crossing points x_k.

The following lemma gives two related criteria for the nonembeddability of pseudoconcave ends.

Lemma 2 i) Let X be a complex cobordism 2-manifold with $\partial X = M_1 \amalg M_0$, M_1 the pseudoconvex end and M_0 the pseudoconcave end. If X contains a compact positive curve $C \subset X$, then M_0 is not embeddable.

ii) Let W be a pseudoconcave surface containing two disjoint positive curves C and C'. Then the compact strictly pseudoconvex CR-3-manifold $M = \partial W$ is not embeddable.

Proof i) If M_0 is embeddable, then M_0 is fillable, i.e. $M_0 = \partial S$ where S is a normal Stein surface [HaLa75]. Let X' be the complex space obtained by gluing S to X along M_0, $X' = X \amalg_{M_0} S$. The interior of X', $\text{Int}(X')$, is a strongly pseudoconvex domain and therefore by [Gr62] $\text{Int}(X')$ is holomorphic convex. That is, $\text{Int}(X')$ has a proper map to a Stein space S', $r : \text{Int}(X') \to S'$. Since S' can not be a point by the properness of the map r, it follows that $\text{Int}(X')$ has nonconstant holomorphic functions. On the other hand, $\text{Int}(X')$ contains a pseudoconcave neighborhood V of the positive curve C (i.e V is a compact neighborhood C which is a pseudoconcave manifold) and therefore all holomorphic functions on the $\text{Int}(X)$ must be constant . This follows from the maximum principle and the fact that the pseudoconcavity of V implies that the set $\hat{V} = \{x \in \text{Int}(X') | |f(x)| \le \sup_{y \in V} |f(y)|$ for all holomorphic functions on $\text{Int}(X')\}$ satisfies $V \subset \text{Int}(\hat{V})$ [AnGr62].

ii) Let V be a pseudoconcave neighborhood of C, then $X = W \setminus Int(V)$ is a complex cobordism manifold with $M = \partial W$ as its pseudoconcancave end. Since X contains the positive curve C', by i) M is not embeddable.

The previous lemmas plus the construction of pseudoconcave surfaces containing disjoint positive curves will give the proof of the main theorem:

Theorem 1 There are complex cobordant 3-dimensional strictly pseudoconvex CR-manifolds M_1 and M_0, with one embeddable and the other not. Equivalently, the embeddability of CR-3-manifolds is not a complex cobordism invariant.

Proof: First, we describe a process to construct a pseudoconcave surface W containing two positive curves C and C' that do not intersect.

Let C_1, C_2 and C_3 be 3 positive curves in the smooth projective surfaces X_1, X_2 and X_3 respectively. First, we glue a neighborhood of C_1 in X_1 with a neighborhood of C_2 in X_2 obtaining a surface Y containing C_1 and C_2, with equivalent neighborhood germs, intersecting transversely at one point.

To do this pick a point $p_1 \in C_1$ and let U_1 be an open subset of X_1 containing p_1 and with a biholomorphism $\phi : U_1 \to \Delta \times \Delta$, Δ the disc of radius 1, such that $\phi(C_1 \cap U_1) = 0 \times \Delta$. Also, let U_2 be an open subset of X_1 such that $C_1 \subset U_1 \cup U_2$ and $\phi(U_1 \cap U_2) \subset \Delta_{\frac{1}{2}} \times (\Delta \setminus \Delta_{\frac{1}{2}})$, $\Delta_{\frac{1}{2}}$ the disc of radius $1/2$. Do the same for C_2 in X_2. Pick a point $p_2 \in C_2$, an open subset $V_1 \subset X_2$ and a biholomorphism $\varphi : V_1 \to \Delta \times \Delta$ such that $\varphi(C_2 \cap V_1) = \Delta \times 0$. Pick V_2 such that $C_2 \subset V_1 \cup V_2$ and $\varphi(V_1 \cap V_2) \subset (\Delta \setminus \Delta_{\frac{1}{2}}) \times \Delta_{\frac{1}{2}}$. The surface Y is the surface given by the open sets $\Delta \times \Delta$, U_2 and V_2 with the gluings of U_2 and V_2 with $\Delta \times \Delta$ given by respectively $\phi|_{U_1 \cap U_2}$ and $\varphi|_{V_1 \cap V_2}$.

Repeat the same gluing argument for neighborhoods of the curve $C_1 \cup C_2$ in Y and the curve C_3 in X_3 (picking a different crossing point). The resulting surface Y' contains the normal crossing divisor $\tilde{C} = C_1 \cup C_2 \cup C_3$ and the neighborhood germs of C_1, C_2 and C_3 in Y' are equivalent to the neighborhood germs of C_1, C_2 and C_3 in X_1, X_2 and X_3 respectively. The surface Y' and the curves C_1, C_2 and C_3 satisfy the conditions of Lemma 1, hence there is a pseudoconcave neighborhood V of $\tilde{C} = C_1 \cup C_2 \cup C_3$ on Y'. The desired pseudoconcave surface W and disjoint positive curves C and C' will be respectively V, C_1 and C_3.

Consider the cobordism manifold $X = W \setminus Int(V_1)$, where V_1 is a pseudoconcave neighborhood of C in $W \cap (U_1 \cup U_2)$. The pseudoconcave end $M_0 = \partial W$ is nonembeddable by Lemma 2. The pseudoconvex end $M_1 = \partial V_1$ is embeddable since it is fillable, V_1 is also a pseudoconcave neighborhood of C_1 in the compact surface X_1.

A pseudoconcave manifold Y is *embeddable* if there is an embedding of Y in a compact complex manifold without boundary X. The problem of preservation of embeddability by complex cobordisms is equivalent to the question of whether a pseudoconcave extension Y of an embeddable pseudoconcave manifold Y' is also embeddable. X is embeddable if and only if ∂Y is fillable, or equivalentely the complex cobordism $Y \setminus \mathrm{Int}(Y')$ preserves embeddalility of its ends. Hence it follows from Theorem 1:

Corollary 1 There is a nonembeddable pseudoconcave surface Y containing inside an embeddable pseudoconcave surface Y'.

In the proof of the theorem, we constructed a pseudoconcave surface W with two disjoint positive curves (repeating the construction ad libitum we could have produced as many disjoint positive curves as desired). If we take out from W a sufficiently small pseudoconcave neighborhood of each of the disjoint positive curves, we obtain:

Corollary 2 There are complex cobordism manifolds X of dimension 2 with more than one pseudoconvex end (impossible in higher dimensions).

Next we consider the problem of whether a CR-function on the pseudoconvex end of a complex manifold with boundary X is the boundary value of a holomorphic function on X. We are interested in the case where all the boundary components of X are strictly pseudoconvex CR-manifolds. The relevance of this boundary value problem comes from the fact that if X has dimension greater than 2 then all CR-functions on the pseudoconvex end are boundary values of a holomorphic function on X. The embeddability of all (2n-1)-dimensional strictly

pseudoconvex CR-manifolds for $n > 2$ implies that all pseudoconcave ends of X are fillable. Hence X is an open subset of a filling (pseudoconvex) of its pseudoconvex end. In [KoRo65] it is shown that all CR-functions on the smooth boundary of a strictly pseudoconvex domain are extendable to holomorphic functions on the domain.

Proposition 1 The extendability of CR-functions on the pseudoconvex end of a complex cobordism manifold X, whose ends are all strictly pseudoconvex CR-manifolds, to holomorphic functions on X does not need to hold if dimension of X is 2.

Proof: The cobordism manifold X described in the proof of the theorem implies the Proposition. Its pseudoconvex end M_1 is embeddable and therefore it has many nonconstant CR-functions and since X contains a positive curve inside (C_3 in the example) all global holomorphic functions on X must be constant as in Lemma 2 i).

SECTION 3

The examples of complex cobordism manifolds described in the proof of Theorem 1 are very different from the complex cobordism manifolds that preserve embeddability. The results in this section go in the opposite direction. A complex cobordism manifold X preserving embeddability must satisfy the condition that all CR-functions on the pseudoconvex end M_1 extend to holomorphic functions on X. As mentioned earlier, this condition also played an essential role in the proof of the preservation of embeddability for strictly CR-cobordisms [EpHe01]. Our examples of complex cobordisms above, where embeddability is not preserved, strongly violated the extendability of CR-functions. Theorem 2 shows that the extendability of CR-functions on the pseudoconvex end is necessary but not sufficient for a complex cobordism to preserve embeddability. Later in this section, in Theorem 3, it is shown that there are small deformations of a complex cobordism 2-manifold preserving embeddability that no longer preserve embeddability.

Theorem 2 There are complex cobordism 2-manifolds X whose pseudoconvex end M_1 is embeddable and all CR-functions on M_1 extend to holomorphic functions on $\mathrm{Int}(X)$, yet the pseudoconcave end M_0 is not embeddable.

Proof Consider three linear $\mathbb{P}^1 \subset \mathbb{P}^2$ C_1, C_2 and C_3 such that $C = C_1 \cup C_2 \cup C_3$ is a normal crossing divisor. Let U_1, U_2 and U_3 be tubular neighborhoods for respectively C_1, C_2 and C_3 such that $U_1 \cap U_2 \cap U_3 = \emptyset$. Let W' be the complex surface that results from $U_1 \cup U_2 \cup U_3 \subset \mathbb{P}^2$ by separating U_2 from U_3. W' has a natural holomorphic map to $\mathbb{P}^2$, $p : W' \to \mathbb{P}^2$ which identifies the points that were separated.
The surface W' and curves in C_1', C_2' and C_3' in W' coming from the three linear $\mathbb{P}^1$, C_1, C_2 and C_3, in $\mathbb{P}^2$ satisfy the conditions stated in Lemma 1 and $C_2' \cap C_3' = \emptyset$. Lemma 1 implies that there is a pseudoconcave surface $W \subset W'$ containing the curve $C_1' \cup C_2' \cup C_3'$. Since W contains 2 disjoint positive curves C_2 and C_3, ∂W can not be embeddable by Lemma 2 ii).

After making the identification of $\mathbb{P}^2 \setminus C_1$ with $\mathbb{C}^2$, consider a sufficiently large ball $B \in \mathbb{C}^2$ such that $\mathbb{P}^2 \setminus B$ is contained in U_1. It follows that $\mathbb{P}^2 \setminus B$ can also be considered as a subset of W'. If one considers an even larger ball B', one can have $(\mathbb{P}^2 \setminus B') \subset W$. The pseudoconcave surface $V = \mathbb{P}^2 \setminus B'$ is naturally embedded as neighborhood of C_1 in $\mathbb{P}^2$ or as a neighborhood of C_1' in W. The open subset $X = W \setminus \text{Int}(V)$ is a complex cobordism with an embeddable pseudoconvex end $M_1 = \partial V$, isomorphic to the boundary of the ball $B' \subset \mathbb{C}^2$, and a nonembeddable pseudoconcave end $M_0 = \partial W$.

We conclude by showing that all CR-functions on M_1 extend to holomorphic functions of $\text{Int}(X)$. The map $p : W' \to \mathbb{P}^2$ gives a CR-isomorphism of M_1 with $\partial B'$ and a holomorphic map $p|_X : X \to B'$. Any CR-function, φ, on M_1 is the pullback by p of a CR-function, φ', on $\partial B'$. The holomophic function on X extending the CR-function, φ, on M_1 will be the pullback by p of the holomorphic function on the ball B' extending φ'.

In order to express a consequence of Theorem 2, we introduce two weakened versions of embeddability of a pseudoconcave surface X_- described in [EpHe2-00].

Let X_- be a pseudoconcave surface with a positive curve Z. The pair (X_-, Z) is called *weakly embeddable* if the holomorphic map defined by the sections of $H^0(X_-, \mathcal{O}(dZ))$, $\varphi_d : X_- \to \mathbb{P}^n$, is injective in some neighborhood of Z for sufficiently large d. The pair (X_-, Z) is called *almost embeddable* if in addition the map $\varphi_d : X_- \to \mathbb{P}^n$ is an embedding outside a proper analytic subset of $X_- \setminus Z$. In [EpHe2-00] the authors ask about the relationship between these two notions of weakened embeddability. The interest sprouted from the fact that one has an operative numerical criterion involving the dimensions $h^0(X_-, \mathcal{O}(dZ))$ characterizing weak embeddability and that almost embeddability of X_- implies embeddability of ∂X_-. In the proof of Theorem 2 we constructed the weakly embeddable pair (W, C_1) such that for any d the maps $\varphi_d : W \to \mathbb{P}^n$ can not distinguish any pair points in W that came from the same point in $U_2 \cap U_3$ and we have:

Corollary 3 The weak embeddability of a pseudoconcave surface (X_-, Z) does not imply that (X_-, Z) is almost embeddable.

The following result states that for a complex cobordism manifold the property of preserving embeddability is not stable under small deformations. We need to recall some facts about the moduli spaces holomorphic neighborhood retracts (HNR) of a curve on a surface for our proof of Theorem 3.

An embedding of a curve C in a manifold X is a HNR if there is a neighborhood W of C with a holomorphic map $r : W \to C$ such that $r|_C = \text{Id}_C$. The space of nonequivalent HNR embeddings of a curve C with normal bundle a prescribed line bundle L is described in [MoRo81]. Let $\{U, V\}$ be a covering of C such that U is a disc and $V \cap U$ is an annulus. A HNR embedding of C with normal bundle L can be constructed by glueing $U \times \mathbb{C}$ with $V \times \mathbb{C}$ via the map $\phi = (\text{Id}, f) : U \cap V \times \mathbb{C} \to U \cap V \times \mathbb{C}$,

$$f(z, u) = g(z)(u + \sum_{i \geq 1} f_i(z)u^{i+1})$$

where z is the coordinate of $U \cap V \subset U$ and $g, f_i \in \mathcal{O}(\mathcal{U} \cap \mathcal{V})$, g is the transition function for the normal bundle L. To the HNR described above we can associate a point in $E(\bigoplus_{i>0} H^1(C, L^{-i}))$, the weighted projective space with the $\mathbb{C}^*$ action $s.(x_1, x_2, , ..., x_n, ...) = (sx_1, s^2 x_2, ..., s^n x_n, ...)$. This is done by considering the f_i as cocycles representing classes of $H^1(C, L^{-i}) \simeq \mathcal{O}(U \cap V)/\delta(\mathcal{O}(V) \oplus \mathcal{O}(U))$. [MoRo81] shows that the neighborhood germs of two HNRs of C with normal bundle L are equivalent only if the corresponding points in $E(\bigoplus_{i>0} H^1(C, L^{-i}))$ coincide.

Theorem 3 There are holomorphic families of complex cobordism 2-manifolds where all the members have their pseudoconvex ends embeddable, but only the central member has the pseudoconcave end embeddable.

Proof We will construct a family of open surfaces such that: 1) each member contains two $\mathbb{P}^1$'s , C_1 and C_2, with normal bundle $\mathcal{O}(1)$ intersecting at one point; 2) only one member has the neighborhood germ of C_1 union with C_2 equivalent to the neighborhood germ of the union of two linear $\mathbb{P}^1$'s in $\mathbb{P}^2$.

Let C_1 and C_2 be two distinct linear $\mathbb{P}^1$'s in $\mathbb{P}^2$ and $\{x_0\} = C_1 \cap C_2$. Let U_0, U_1 and U_2 be a covering of a neighborhood U of $C = C_1 \cup C_2$ such that $x_0 \in U_0$, $C_1 \subset U_0 \cup U_1$, $C_2 \subset U_0 \cup U_2$, and $U_1 \cap U_2 = \emptyset$. Since the neighborhood germs of a linear $\mathbb{P}^1$ in $\mathbb{P}^2$ and of the zero section of the line bundle $\mathcal{O}(1)$ over $\mathbb{P}^1$ are equivalent, we can choose U_0 and U_2 to be biholomorphic to the bidisc, $U_0 \cap C_2 = \Delta \times 0$, $U_2 \cap C_2 = \Delta \times 0$ and $U_0 \cap U_2 \subset (\Delta \setminus \Delta_{\frac{1}{2}}) \times \Delta \subset U_0$. Moreover, we can describe $U_0 \cup U_2$ as the complex surface resulting from glueing U_0 and U_2 via the map $\phi = (\text{Id}, f)(z, u) : (U_0 \cap U_2 \cap C_2) \times \Delta \to (U_0 \cap U_2 \cap C_2) \times \Delta$, where $f(z, u) = g(z)u$ and g is the transition function for the line bundle $\mathcal{O}(1)$.

Change the glueing map of U_0 and U_2 by varying $f(z, u)$ holomorphically with respect to a parameter t, $f_t(z, u)$, by introducing nontrivial higher order terms $f_{i,t}$, $i > 1$. From the description in the paragraph before the theorem, it follows that by making $f_{i,t} \in H^1(\mathbb{P}^1, \mathcal{O}(1)) \neq 0$ for $t \neq 0$ the resulting holomorphic family of HNR of $\mathbb{P}^1$ is such that only for $t = 0$ the HNR is equivalent to the linear $\mathbb{P}^1$ in $\mathbb{P}^2$.

Let $\omega : \mathcal{V} \to \Delta$ be the family of surfaces , with $\mathcal{V}_t = V_t$ and $V_0 = U$, obtained by keeping the gluing of U_0 with U_1 but changing the gluing of U_0 with U_2 as in the previous paragraph. Each member V_t of the family has the curve $C = C_1 \cup C_2$ embedded with the same normal bundle. Hence the germ of the tubular neighborhood of C in all the V_t is diffeomorphic to the germ of the tubular neighborhood T of C in V_0. It follows that there is a smooth map $\phi : T \times \Delta \to \mathcal{V}$, with each $\phi_t : T \to T_t = \phi(T \times t) \subset V_t$ a diffeomorphism from T to a tubular neighborhood of C in V_t sending C in T to C in V_t. The union of all T_t, $\mathcal{T} = \bigcup_{t \in \Delta} T_t \subset \mathcal{V}$, gives a family of surfaces $\omega|_{\mathcal{T}} : \mathcal{T} \to \Delta$ (the family $\mathcal{T}$ can also be perceived as the variation of the complex structure induced by the diffeomorphisms ϕ_t on differential manifold T).

By Lemma 1 there is a strictly plurisubharmonic function $g : V_0 \setminus C \to \mathbb{R}$ such that $\{x \in V_0 : g(x) \geq c\} \subset T$ for $c \gg 0$ and $S_c = \{x \in T : g(x) = c\}$ is a compact SPCR-3-manifold. After possibly shrinking Δ, $c \gg 0$ can be such that $g_t = g \circ \phi_t^{-1} : T_t \setminus C \to \mathbb{R}$ is a

strictly plurisubharmonic on a neighborhood of $\phi_t(S_c)$ for all $t \in \Delta$. Let $Y_t = \{x \in T_t | g_t(x) > c$ or $x \in C\} \subset T_t$, the union of all Y_t, $\mathcal{Y} = \bigcup_{t \in \Delta} Y_t \subset \mathcal{V}$, gives a family, $\omega|_{\mathcal{Y}} : \mathcal{Y} \to \Delta$, of pseudoconcave neighborhoods of $C = C_1 \cup C_2$.

We claim that the boundaries, ∂Y_t, of the pseudoconcave surfaces Y_t are not embeddable for $t \neq 0$. By construction, Y_t contains the rational curve C_2 with normal bundle of degree one but whose embedding is not equivalent to the embedding of the linear $\mathbb{P}^1 \subset \mathbb{P}^2$ for $t \neq 0$. If a ∂Y_t were embeddable for $t \neq 0$ then Y_t would be contained in a smooth algebraic surface S_t coming from filling ∂Y_t and resolving the singularities. The claim follows from the result 6.13 of [MoRo81], which proves that the only neighborhood germ of $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1)$ that can be embedded in an algebraic surface is the neighborhood germ of the linear $\mathbb{P}^1 \subset \mathbb{P}^2$.

To construct the family of complex cobordisms, we notice that all Y_t contain a pseudo-concave neighborhood of C_1 whose boundary is embeddable. By Lemma 1, we can choose a sufficiently small pseudoconcave neighborhood $W \subset V_0$ of C_1 such that $W_t = \phi_t(W)$ gives a family of pseudoconcave surfaces, $\omega|_{\mathcal{W}} : \mathcal{W} \to \Delta$. All members of the family $\mathcal{W}$ are biholo-morphic to the pseudoconcave neighborhood $W \subset \mathbb{P}^2$ of C_1, therefore their pseudoconcave ends $\partial W_t \simeq \partial W$ are embeddable.

The family of complex cobordism manifolds $\omega|_{\mathcal{X}} : \mathcal{X} = \mathcal{Y} \setminus \text{Int}(\mathcal{W}) \to \Delta$ gives the desired example. The pseudoconvex end $M_{0_t} = \partial W_t$ of X_t is CR-isomorphic to the embeddable SPCR-3-manifold ∂W but the pseudoconcave end $M_{1_t} = \partial Y_t$ is not embeddable for $t \neq 0$, as was explained above.

Remark: the example in the proof also shows that: the extendability of the CR-functions of the pseudoconvex end to holomorphic functions on the complex-cobordism manifold is a property that is not locally stable in families of complex-cobordisms. Suppose all CR-functions on ∂W_t extend to holomorphic functions on X_t, then the natural fixed inclusion $W_t \subset \mathbb{P}^2$ would extend to holomorphic maps $i_t : Y_t \to \mathbb{P}^2$. For $t \neq 0$ sufficiently small, i_t is a small perturbation of i_0. Therefore CR-map of ∂Y_t to $\mathbb{C}^2$ induced by i_t is an embedding since it is a small peturbation of CR-embedding of ∂Y_0 in $\mathbb{C}^2$, which gives the desired contradiction.

It is well known that the embeddability of strictly pseudoconvex CR-structures on a 3-manifold is not stable under small deformations of the CR-structure. It follows from the proof of the theorem that the instability of the embeddability property still holds under the condition that all deformed CR-structures are complex cobordant to a fixed embeddable SPCR-3-manifold.

Corollary 4 Let M_1 be an embeddable SPCR-3-manifold. The embeddability of SPCR-3-manifolds complex-cobordant to M_1 is not stable, for small deformations of the CR-structure preserving the property of being complex-cobordant to M_1.

Proof In the proof of theorem 3 all the SPCR-3-manifolds M_{1t} are complex cobordant to S^3 with the canonical embeddable CR-structure.

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