HOLOMORPHIC FUNCTIONS AND VECTOR BUNDLES ON COVERINGS OF PROJECTIVE VARIETIES

Fedor Bogomolov* Bruno De Oliveira**

* Courant Institute for Mathematical Sciences **University of Miami

Abstract. Let X be a projective manifold, $\rho : \tilde{X} \to X$ its universal covering and $\rho^* : Vect(X) \to Vect(\tilde{X})$ the pullback map for the isomorphism classes of vector bundles. This article establishes a connection between the properties of the pullback map ρ^* and the properties of the function theory on $\tilde{X}$. We prove the following pivotal result: if a universal cover of a projective variety has no nonconstant holomorphic functions then the pullback map ρ^* is almost an imbedding.

0. Introduction

It is still unknown whether the non-compact universal covers $\tilde{X}$, $\rho : \tilde{X} \to X$, of projective varieties X must have nonconstant holomorphic functions. The existence of holomorphic functions on $\tilde{X}$ can usually be translated into geometric properties of the complex manifolds X and $\tilde{X}$. An example of this phenomenon is the Shafarevich conjecture: the universal cover $\tilde{X}$ of a projective variety X is holomorphically convex. The Shafarevich conjecture asks for an abundance of holomorphic functions on $\tilde{X}$ which imply precise geometric properties on X (see [Ko93], [Ca94] and [Ka95]). On this article we give new approach to the problem of the existence of holomorphic functions on universal covers. More precisely, the article establishes the relation between the existence of holomorphic functions on universal covers $\tilde{X}$, $\rho : \tilde{X} \to X$, and the identification on $\tilde{X}$ of the pullback of distinct isomorphism classes of vector bundles on X.

The known paths to the production of holomorphic functions on $\tilde{X}$ involve the construction of closed holomorphic 1-forms or exhaustion functions with plurisubharmonic properties on $\tilde{X}$. The construction of the desired closed (1,0)-forms or exhaustion functions on $\tilde{X}$ involve the following methods: (a) properties of the fundamental group $\pi_1(X)$ in combination with Hodge theory and non-abelian Hodge theory (see [Si88], [Ka95] and [Ey04]) for the most recent results and references); (b) curvature properties of X (see for

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example [SiYa77] and [GrWu77]), (c) explicit descriptions of X (see for example [Gu87] and [Na90]). None of these methods are at the moment sufficiently general to provide a nonconstant holomorphic function for the universal cover of an arbitrary projective variety.

Our approach to the existence of holomorphic functions on $\tilde{X}$ is different. We connect the existence of nonconstant holomorphic functions on $\tilde{X}$ to properties of $\rho^* : Vect(X) \to Vect(\tilde{X})$, the pullback map for vector bundles. An extreme example of this connection is the case where ρ^* identifies all isomorphism classes corresponding to deformation equivalent bundles. Let X be a projective manifold such that the pullback map identifies all isomorphism classes of holomorphic vector bundles on X that are isomorphic as topological bundles. Then $\tilde{X}$ must be Stein, see observation 3.1.

In previous work [BoDeO04] and [DeO04], the authors investigated how the existence of nontrivial cocycles $\alpha \in H^1(X, V)$ such that $\rho^*\alpha = 0$ imply strong convexity properties of the universal cover $\tilde{X}$. In that work the vector bundle V had to have strong negative or semi-negative properties. In section 2, we modify our previous approach and are able to obtain holomorphic functions under very weak negativity properties of V (see for example proposition 2.7). This article is a consequence of the nice fact that the bundle $EndV$ is well suited to our new approach.

Our strategy is to identify the properties of the pullback map ρ^* that are implied by the absence of nonconstant holomorphic functions on $\tilde{X}$. We are able to obtain holomorphic functions on $\tilde{X}$ from cocycles $\alpha \in H^1(X, EndV)$ such that $\rho^*\alpha = 0$ if V is absolutely stable (see section 1 and below). It is well known that the vector space $H^1(X, EndV)$ has a geometrical meaning, it is the tangent space at V to moduli space of holomorphic vector bundles on X topologically equivalent to V. As a consequence, it follows that if the fibers of the pullback map ρ^* are not zero dimensional then $\tilde{X}$ must have nonconstant holomorphic functions. The main theorem of this article is the application of our results to the case where $\tilde{X}$ has only constant holomorphic functions.

Theorem 3.10. *Let X be a projective manifold whose universal cover has only constant holomorphic functions. Then:*

a) The pullback map $\rho_0^* : Mod_0(X) \to Vect(\tilde{X})$ is a local embedding ($Mod_0(X)$ is the moduli space of absolutely stable bundles).

b) For any absolutely stable bundle E there are only finite number of bundles F with $\rho^*E = \rho^*F$.

c) The absolute stable vector bundle E determines a finite unramified cover $p : X' \to X$ of degree $d \leq rkE!$. On X' there is a fixed collection of π_1-simple vector bundles $\{E'_i\}_{i=1,\dots,m}$ such that a vector bundle F on X satisfies $\rho^*F \simeq \rho^*E$ if and only if:

$$p^*F = E'_1 \otimes \mathcal{O}(\tau_1) \oplus \dots \oplus E'_1 \otimes \mathcal{O}(\tau_m)$$

The bundles $\mathcal{O}(\tau_i)$ are flat bundles associated with finite linear representations of $\pi_1(X')$ of a fixed rank k with $rkE|k$.

A vector bundle V is absolutely stable if for any coherent subsheaf $\mathcal{F} \subset V$ with rk$\mathcal{F}$ < rk V a multiple of the line bundle (rk Vdet$\mathcal{F}$ − rk$\mathcal{F}$det V)* can be represented by a nonzero effective divisor. In particular, absolutely stable bundles are stable with respect to all polarizations of X. For projective surfaces absolutely stable bundles are exactly the bundles that are stable with respect to all elements in the closure of the polarization cone.

The main theorem imposes strong geometric constraints on the pullback map ρ^* if $\tilde{X}$ is not to have nonconstant holomorphic functions. In particular, it says that the pullback map should be almost an embedding. The authors believe that this imposition on the pullback map ρ^* should not hold for any projective variety (see the remarks at the end of section 3). If the authors are correct then theorem 3.10 can be instrumental in the proof of the existence of nonconstant holomorphic functions on non-compact universal covers of projective varieties.

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1. Stability background and absolute stability

There are several notions of stability for vector bundles on projective varieties. Each stability condition is better suited to tackle a class of problems. We use the Mumford-Takemoto H-stability, see below, but we are specially interested in a stronger stability condition called absolute stability, see definition 1.1. The H-stability allows one to obtain a good parameterizing scheme for vector bundles on a projective variety X. There is an algebraic parameterization for H-stable bundles with given topological invariants. This parameterization space has all the basic properties of a coarse moduli space (see for example [HuLe97], [Ma77]). As it will be seen below, an absolutely stable bundle is stable with respect to all polarizations. On the other hand, on a surface X a bundle which is H-stable for all divisors H in the nef cone of X will be absolutely stable.

Let $N^1_{\mathbb{R}}(X)$ be the finite dimensional $\mathbb{R}$-vector space consisting of the set of numerical equivalence classes of $\mathbb{R}$-Cartier divisors on X. Let $P(X) \subset N^1_{\mathbb{R}}(X) \hookrightarrow H^2(X,\mathbb{R}) \cap H^{1,1}(X,\mathbb{C})$ be the polarization cone of X containing the ample classes. Denote its closure in $N^1_{\mathbb{R}}(X)$ by $\overline{P(X)}$ ($\overline{P(X)} = Nef(X)$ the convex cone of nef divisors).

Let E be a vector bundle on a projective variety X of dimension n and H be nef divisor. E is said to be $H-semistable$ if the inequality (rk E$det\mathcal{F}$ − rk$\mathcal{F}detE$)$.H^{n-1} \leq 0$ holds for all coherent subsheaves $\mathcal{F} \subset E$. Moreover, if for all coherent subsheaves $\mathcal{F} \subset E$ of lower rank (rk E$det\mathcal{F}$ − rk $\mathcal{F}detE$)$.H^{n-1} < 0$ holds then E is said to be $H-stable$. The vector bundle E is $H-unstable$ if it has an H-destabilizing subsheaf $\mathcal{F}$, i.e there is a coherent subsheaf $\mathcal{F}$ with $0 < \text{rk}\mathcal{F} < \text{rk E}$ such that $(\text{rk E}det\mathcal{F} - \text{rk}\mathcal{F}detE).H^{n-1} > 0$ holds. The number $\mu_H(\mathcal{F}) = (detF/\text{rk}\mathcal{F}).H^{n-1}$ is called the $H-slope$ of F. H-stability of E is equivalent to the fact that any coherent subsheaf of E with smaller rank has a smaller H-slope than E.

We introduce the following notion of stability:

Definition 1.1. *Let $N_{eff}(X) \subset N^1_{\mathbb{Q}}(X)$ be the $\mathbb{Q}$-convex cone spanned by the classes of effective $\mathbb{Q}$-divisors. A vector bundle E is absolutely stable if for any coherent subsheaf $\mathcal{F} \subset E$ with $rk\mathcal{F} < rk\ E$ the following holds: $rk\ Edet\mathcal{F} - rk\mathcal{F}det\ E \in -N_{eff}(X)^+ = -N_{eff}(X) \setminus 0$.*

The absolutely stable bundles are stable with respect to all polarizations. Later in this article, we will use the following consequence of the definition: if X is regular, i.e. $Pic^0(X) = 0$, and E is absolutely stable then the line bundles $\text{rk Edet}\mathcal{F} - \text{rk}\mathcal{F}\text{det E}$ will have a multiple with a nontrivial section.

The notion of H-stability for H is the same as for aH, $a \in \mathbb{R}^+$. So, we can talk of stability with respect to elements on the projectivization $\mathbb{P}(Nef(X)) \subset \mathbb{P}(N^1_{\mathbb{R}}(X))$. The base of the projectivization of the nef cone $\mathbb{P}(Nef(X))$ is compact ($Nef(X)$ is closed in $N^1_{\mathbb{R}}(X)$). Hence the notion of a stable bundle with respect to the nef cone is well defined (E is stable with respect to the nef cone if it is H-stable for all nef divisors H).

Lemma 1.2. *Let E be a vector bundle over projective surface X which is stable with respect to the nef cone. Then E is absolutely stable.*

Proof. The Kleiman duality for surfaces states that

$$\overline{N_{eff}(X)} = \{\gamma \in N^1_{\mathbb{R}}(X) \mid (\delta.\gamma) \geq 0, \forall \delta \in Nef(X)\}$$

Since the cone $\mathbb{P}(Nef(X))$ is compact, it follows that:

$$\{\gamma \in N^1_{\mathbb{R}}(X) \mid (\delta.\gamma) > 0, \forall \delta \in Nef(X)\} \subset \text{Int}N_{eff}(X)$$

From the above, it follows immediately that if E is stable with respect to the nef cone then $(\text{rk}\mathcal{F}\text{det E} - \text{rk Edet}\mathcal{F}) \in \text{Int}N_{eff}(X) \subset N_{eff}(X)^+$ for all coherent subsheaves $\mathcal{F} \subset E$ with $\text{rk}\mathcal{F} < \text{rk E}$. Thus E is absolutely stable.

For a general projective variety X the stability property is captured on surfaces which are complete intersections in the initial variety.

The condition of absolute stability is the right condition for the formulation of the results of the next sections. We will need to describe some properties of the bundle End$E$ for an H-stable bundle E. We will also need results on the theory of stable bundles for smooth projective curves and on how stability behaves under restriction maps. We start with the basic result:

Lemma 1.3. *Let E be a vector bundle on X which is stable with respect to some $H \in P(X)$ and $End_0(X)$ the sheaf of traceless endomorphisms of E. Then $H^0(X, End_0 E) = 0$.*

For a proof see, for example, chapter 1 of [HuLe97]).

If C is a smooth curve and E is a stable vector bundle over C then by a classical result of Narasimhan-Seshadri $End_0 E$ is obtained from a unitary representation τ of the fundamental group $\pi_1(C)$ in $PSU(n), n = rkE$. The elements of $PSU(n)$ act on the matrices in $\mathrm{End}\mathbb{C}^n$ by conjugation. Since the bundle E is stable the representation τ is irreducible.

Lemma 1.4. *If E is a stable vector bundle over a smooth projective curve C then the bundle $EndE$ is a direct sum of stable vector bundles of degree 0.*

Proof. This fact is well known and the decomposition into a direct sum of stable bundles corresponds to the decomposition of the unitary representation of $\pi_1(X)$ in $PSU(n) \subset SU(n^2 - 1)$ under the above imbedding. □

Let E be a vector bundle over a projective variety X. Let $C_E \subset N^1_{\mathbb{R}}(X)$ be the cone generated by the classes $det\mathcal{L}$, where the $\mathcal{L}$'s are the rank 1 coherent subsheaves of $\bigcup_{n=0}^{\infty} (EndE)^{\otimes n}$. Let us recall the following result from [Bo94] which follows from invariant theory (the notation $Span^+$ used below means finite linear combinations with coefficients ≥ 0):

Lemma 1.5. *i) The cone $C_E = Span^+\{det\mathcal{L} \in N^1_{\mathbb{R}}(X) \mid \mathcal{L} \subset \bigcup_{n=0}^{\infty}(EndE)^{\otimes n}$ and $rk\mathcal{L} = 1\}$ is also generated by classes of the form $det\mathcal{F} - (rk\mathcal{F}/rkE)detE$, where $\mathcal{F} \subset E$ are proper coherent subsheaves of E and a finite collection $D_1, ..., D_k \in -N_{eff}(X)$.*

ii) For any bundle E of rank k there is a natural reductive structure group $G_E \subset GL(k)$ of E such that C_E is generated by the line subbundles L corresponding to the characters of parabolic subgroups in G_E.

The group G_E is defined modulo scalars by the set of subbundles $L \in (EndE)^{\otimes n}$ for all n with $c_1(L) = 0$. If $G_E = GL(k), SL(k)$ then the line subbundles L are exactly the line bundles $det\mathcal{F} - (rk\mathcal{F}_i/rkE)detE$. However, if the group G_E is smaller than above then the line bundles generating C_E correspond to determinants of special subsheaves of E.

Corollary 1.6. *i) Let E be an absolutely stable vector bundle on a smooth projective variety X and $\mathcal{A} \subset EndE$ be a coherent subsheaf. Then $det\mathcal{A} \in -N_{eff}(X)$.*

ii) If E is absolutely stable and $E = E' \otimes F$ then both E', F are absolutely stable since the corresponding parabolic group G_E is contained in the group product $G_{E'} \times G_F$ and the cone C_E is a sum $C_{E'} + C_F$.

Proof. The divisor $det\mathcal{A}$ belongs to C_E since $det\mathcal{A} \subset \bigotimes^{\mathrm{rk}(EndE)}(EndE)$. Hence it follows from lemma 1.5 that $det\mathcal{A} = \Sigma a_i(det\mathcal{F}_i - (rk\mathcal{F}_i/rkE)detE) + \Sigma b_j D_j$ where $a_i, b_j \geq 0$ with $\mathcal{F}_i$ coherent subsheaves of E and $D_j \in -N_{eff}(X)$. Thus i) follows from the condition of absolute stability, i.e all elements $det\mathcal{F}_i - detE(rk\mathcal{F}_i/rkE)$ belong to $-N_{eff}(X)$.

The item ii) is a consequence of the fact that the parabolic subgroups in $G_{E'} \times G_F$ are products of the parabolic subgroups in $G_{E'}, G_F$. □

Many properties of H-stable bundles on arbitrary projective varieties can be derived from their restrictions on smooth curves. This is manifested in the following two results that will be important for us later on.

Lemma 1.7. *Let X be a projective variety, H a polarization of X, E an H-stable vector bundle and C a generic curve in kH^{n-1} for $k \gg 0$. Then:*

1) The restriction of E to C is stable.

2) Any saturated coherent subsheaf $\mathcal{F} \subset EndE|_C$ with $\mu_H(\mathcal{F}) = 0$ is a direct summand of $EndE|_C$.

3) The set of saturated subsheaves $\mathcal{F}$ of $EndE$ with $\mu_H(\mathcal{F}) = 0$ coincide via the restriction map with the similar set for $EndE|C$ on C.

4) The bundle $EndE$ is H-semistable and it is a direct sum of H-stable bundles F_i with $\mu_H(F_i) = 0$.

Proof. 1), 2) follows from general results, see for example [Bo78], [Bo94] and [HuLe97].

3) For any given coherent sheaf $\mathcal{S}$ on X there is an isomorphism $H^0(X, \mathcal{S}) \simeq H^0(C, \mathcal{S}|_C)$ if k is sufficiently large. This isomorphism follows from the vanishing of the cohomology of coherent sheaves on projective varieties after being tensored with a sufficient large multiple of an ample line bundle. In particular, there are the restriction isomorphisms $H^0(X, End(EndE)) \simeq H^0(C, End(EndE)|_C)$. A saturated coherent subsheaf $\mathcal{F}_C \subset EndE|_C$ with $\mu_H(\mathcal{F}_C) = 0$ is a direct summand of $EndE|_C$ by 2). Hence $\mathcal{F}_C$ is associated with a projection of $EndE|_C$, $P_C \in H^0(C, End(EndE)|_C)$. The above isomorphism implies that $P_C = P_X|_C$ for a unique $P_X \in H^0(X, End(EndE))$ and that $P_X^2 = P_X$ since $P_X^2|_C = P_C^2 = P_C = P_X|_C$. Therefore the saturated subsheaf $\mathcal{F}_C \subset EndE|_C$ determines a unique direct summand F_X of $EndE$. The sheaf of sections of F_X, $\mathcal{F}_X$, is saturated and satisfies $\mathcal{F}_X|_C = \mathcal{F}_C$ and $\mu_H(\mathcal{F}_X) = det\mathcal{F}_X.H^{n-1}/\mathrm{rk}\mathcal{F}_X = \frac{1}{k}deg(det\mathcal{F}_C)/\mathrm{rk}\mathcal{F}_C = \frac{1}{k}\mu_H(\mathcal{F}_C) = 0$. On the other hand, any saturated subsheaf $\mathcal{F}_X$ of $EndE$ with $\mu_H(\mathcal{F}_X) = 0$ determines by restriction a saturated coherent subsheaf $\mathcal{F}_C = \mathcal{F}_X|_C \subset EndE|_C$ with $\mu_H(\mathcal{F}_C) = 0$.

4) Since vector bundles F on X such that $F|_C$ is stable must be H-stable (C is a curve in kH^{n-1}), then 4) follows from lemma 1.4 and 3). □

Corollary 1.8. *Let E be an absolute stable vector bundle on a projective variety X. Then $EndE = \bigoplus_{i=1}^{l} F_i$, where the F_i are absolute stable bundles with $\mu_H(F_i) = 0$ for any H in $P(X)$.*

Proof. The vector bundle E is H-stable for all polarizations H of X. Fix any polarization H, part 4) of lemma 1.7 implies that $EndE = \bigoplus_{i=1}^{l} F_i$ where all the F_i are H-stable with $\mu_H(F_i) = 0$. Our claim is that the F_i are absolute stable vector bundles.

Let $\mathcal{F}$ be a saturated coherent subsheaf of one of the direct summands F_i with $\mathrm{rk}\mathcal{F} < \mathrm{rk}F_i$. We need to show that $det\mathcal{F} \in -N_{eff}(X)^+$. Lemma 1.6 almost gives the result, $det\mathcal{F} \in -N_{eff}(X)$. To exclude $det\mathcal{F}$ being numerically equivalent to $\mathcal{O}_X$, we notice that part 3) of lemma 1.7 implies that if $\mu_H(\mathcal{F}) = 0$ then $\mathcal{F}$ is a direct summand of $EndE$ and hence also of F_i. This is not possible since F_i is H-stable. □

2 Holomorphic functions via Pullback of extensions and flat bundles

This section presents two approaches that give holomorphic functions on the universal covers of projective varieties. In section 2.1 we describe our new method to produce holomorphic functions on an infinite unramified covering of a projective variety X, $f : X' \to X$. In section 2.2 we collect some results on the relation between flat bundles on X and the existence of holomorphic functions on the universal cover $\tilde{X}$. This two approaches are in some sense orthogonal to each other (see for example proposition 2.7). This complementary relation between the two approaches is essential to our results of section 3.

2.1 Pullback of extensions and holomorphic functions.

Let X be a complex manifold and $\tilde{X}$ its universal cover. In this article we explore the existence of nonconstant holomorphic functions on $\tilde{X}$ coming from the existence of nontrivial extensions of vector bundles on X which pullback to the trivial extension on $\tilde{X}$. Let V a vector bundle of rank r on X. We will use the common abuse of notation where V also denotes the sheaf of sections of V. An extension of $\mathcal{O}_X$ by a vector bundle V is an exact sequence:

$$0 \to V \to V_\alpha \to \mathcal{O}_X \to 0 \tag{2.1}$$

There is a 1-1 natural correspondence between cocycles $\alpha \in H^1(X, V)$ and isomorphism classes of extensions of $\mathcal{O}$ by V. A cocycle α cohomologous to zero corresponds to the trivial extension $V_\alpha = V \oplus \mathcal{O}$. It follows from the above that the existence of a nontrivial extension of $\mathcal{O}_X$ by a vector bundle V that becomes trivial when pulled back to $\tilde{X}$ is the same as the existence of a nontrivial cocycle $\alpha \in H^1(X,V)$ that becomes trivial when pulled back to $\tilde{X}$. The following standard result (see lemma 4.1.14 of [La04]) shows that this is only possible for infinite covers.

Lemma 2.1. *Let $f : Y \to X$ a finite morphism between irreducible normal varieties X and Y and V a vector bundle over X. If $s \in H^1(X,V)$ is nontrivial then $f^*s \in H^1(Y, f^*V)$ is also nontrivial.*

Let $p : X' \to X$ be an infinite unramified Galois covering of a complex manifold X and V a vector bundle over X. The following lemma shows how the existence of a nontrivial cocycles $\alpha \in H^1(X,V)$ such that $p^*\alpha = 0$ implies the existence of nonzero sections of p^*V.

Lemma 2.2. *Let $p : X' \to X$ be an infinite unramified Galois covering of a complex manifold X and V a vector bundle over X. If the kernel $p^* : H^1(X,V) \to H^1(X', p^*V)$ is nontrivial then the vector bundle p^*V on X' has nonzero sections.*

Proof. Let G be the Galois group of the covering $p : X' \to X$ and $\alpha \in H^1(X,V)$ a nontrivial cocycle such that $p^*\alpha = 0$. This implies that there is a non-split extension of the trivial bundle by V:

$$0 \to V \to V_\alpha \to \mathcal{O}_X \to 0 \tag{2.1}$$

that pullbacks to the trivial extension:

$$0 \to p^*V \to p^*V_\alpha \simeq p^*V \oplus \mathcal{O}_{X'} \to \mathcal{O}_{X'} \to 0 \tag{2.2}$$

Any splitting of the pullback extension (2.2) gives a nontrivial section $s \in H^0(X', p^*V_\alpha)$. The Galois group G acts on p^*V_α and hence we also obtain the sections $\gamma s \in H^0(X', p^*V_\alpha)$ for $\gamma \in G$. The sections γs can not all be the same since that would imply that the splitting (2.2) would descend to a splitting of (2.1), which would give a contradiction. Hence there will be nontrivial sections of p^*V_α of the form $s' = \gamma s - \gamma' s$. To finish the lemma we note that $s' \in H^0(X', p^*V)$ since γs and $\gamma' s$ have the same projection in $\mathcal{O}'_X$. □

Definition 2.3. *A sheaf $\mathcal{F}$ on X is (generically) π_1-globally generated if on the universal cover $\rho : \tilde{X} \to X$ the sheaf $\rho^*\mathcal{F}$ is (generically) globally generated.*

Lemma 2.4. *Let V be a vector bundle with a nontrivial cocycle $\alpha \in H^1(X,V)$ such that $\rho^*\alpha = 0$. Then there is an π_1-globally generated coherent subsheaf $\mathcal{F} \subset V$ such that the cocycle α comes from a cocycle $\beta \in H^1(X, \mathcal{F})$.*

Proof. It follows from lemma 2.2 that the vector bundle ρ^*V has nontrivial sections. Let $\tilde{\mathcal{F}}$ be the subsheaf of ρ^*V that is generated by the global sections. Clearly, $\tilde{\mathcal{F}}$ is $\pi_1(X)$-invariant. Let $\mathcal{F}$ be the subsheaf of V whose stalk at $x \in X$ is generated by the germs of the global sections of ρ^*V at one pre-image $\tilde{x} \in \rho^{-1}(x)$. Any choice of pre-image would give the same stalk. The sheaf $\mathcal{F}$ is coherent because of the strong noetherian property of coherent sheaves on complex manifolds. The sheaf $\mathcal{F}$ is the quotient $\tilde{\mathcal{F}}/\pi_1(X)$ hence it is π_1-globally generated.

Let $i_* : H^1(X, \mathcal{F}) \to H^1(X, V)$ and $q_* : H^1(X, V) \to H^1(X, V/\mathcal{F})$ be the morphisms from the cohomology long exact sequence associated with $0 \to \mathcal{F} \to V \to V/\mathcal{F} \to 0$. To show the existence of a cocycle $\beta \in H^1(X, \mathcal{F})$ with $\alpha = i_*\beta$ we need to verify that $q_*\alpha = 0$.

The extension $0 \to V \to V_\alpha \to \mathcal{O}_X \to 0$ associated with the cocycle α induces the exact sequence:

$$0 \to V/\mathcal{F} \to V_\alpha/\mathcal{F} \to \mathcal{O}_X \to 0 \tag{2.3}$$

The triviality of $q_*\alpha$ holds if (2.3) splits. The exact sequence (2.3) is the quotient of the the exact sequence:

$$0 \to \rho^*V/\rho^*\mathcal{F} \to (\rho^*V)_{\rho^*\alpha}/\rho^*\mathcal{F} \to \mathcal{O}_{\tilde{X}} \to 0 \tag{2.4}$$

via the action of $\Gamma = \pi_1(X)$ on $(\rho^*V)_{\rho^*\alpha}$ that gives V_α. The extension of $\mathcal{O}_{\tilde{X}}$ by ρ^*V associated with $\rho^*\alpha$ splits by the hypothesis, but this splitting is not $\pi_1(X)$-invariant. The splitting is given by a section $s \in H^0(\tilde{X}, \rho^*V_\alpha)$, that is not preserved by the $\pi_1(X)$-action. On the other hand, this splitting induces a Γ-invariant splitting of (2.3) since $s - \gamma s \in H^0(\tilde{X}, \rho^*\mathcal{F})$ and $\rho^*\mathcal{F}/\Gamma = \mathcal{F}$. $\square$

Let $\rho^* : Vect(X) \to Vect(\tilde{X})$ be the pullback map sending the set of isomorphism classes of vector bundles on X into the set of isomorphism classes of vector bundles on $\tilde{X}$. The flat vector bundle on X obtained from a linear representation τ of the fundamental group of X or its sheaf of sections is denoted by $\mathcal{O}(\tau)$. By construction $\rho^*\mathcal{O}(\tau)$ is the trivial bundle on the universal covering $\tilde{X}$ with the rank of τ.

The next lemma is a flexible tool to produce holomorphic functions on the universal coverings that will be a key ingredient of our results.

Lemma 2.5. *Let $\mathcal{F}$ be a generically π_1-globally generated coherent torsion free sheaf on a complex manifold X such that $det(\mathcal{F})^{-k}$ has a nontrivial section. Then one of the following holds:*

1) $\tilde{X}$ has a nonconstant holomorphic function.

2) $\mathcal{F}$ is the sheaf of sections of a flat bundle, $\mathcal{F} \cong \mathcal{O}(\tau)$.

Proof. Let s_1,...,s_r be a collection of sections of $\rho^*\mathcal{F}$ generating $\rho^*\mathcal{F}$ generically, where r is the rank of $\mathcal{F}$. From the sections s_1,...,s_r one gets a nontrivial section of $\det(\rho^*\mathcal{F})$ $s = s_1 \wedge ... \wedge s_r \in H^0(\tilde{X}, \det(\rho^*\mathcal{F}))$, the pairing of s^n with a nontrivial section $t \in H^0(\tilde{X},\ \det(\rho^*\mathcal{F})^{-n})$ gives a holomorphic function f on $\tilde{X}$. By hypothesis the function

f is nonzero on a open set of $\tilde{X}$. There are two cases where the function f vanishes at $p \in \tilde{X}$. Case: i) $s_1(p)$,...,$s_r(p)$ do not generate $\rho^*\mathcal{F}_p/m_p\rho^*\mathcal{F}_p$; case ii): $t(p)$ is zero.

Suppose statement 1) does not hold. Then f must be a nonzero constant function, which implies that $s_1(p)$,...,$s_r(p)$ are linear independent at all $p \in \tilde{X}$. Hence the morphism $(s_1, ..., s_r) : \mathcal{O}^r \to \rho^*\mathcal{F}$ induced by the sections is an isomorphism. The nonexistence of holomorphic functions on $\tilde{X}$ implies that all sections of $\rho^*\mathcal{F} = \mathcal{O}^r$ are constant. The linear action of $\pi_1(X)$ on $H^0(\tilde{X}, \mathcal{O}^r) = \mathbb{C}^r$ gives a representation $\tau : \pi_1(X) \to GL(r, \mathbb{C})$ and $\mathcal{F}$ is the sheaf of sections of the flat vector bundle $\tilde{X} \times_\tau \mathbb{C}^r$. □

Corollary 2.6. *Let X be a projective variety such that $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. If E is a vector bundle such that $detE \in -N_{eff}(X)$ then E is not generically π_1-global generated unless $detE$ has finite order in $PicX$ and E is flat.*

The next result is an application of lemma 2.5 for vector bundles.

Proposition 2.7. *Let E be an absolutely stable vector bundle over a projective manifold X with $(detE)^l = \mathcal{O}$ for some $l \in \mathbb{Z}$. If there is a nontrivial cocycle $\alpha \in H^1(X, E)$ such that $\rho^*\alpha = 0$ then one of the following possibilities holds:*
1) $\tilde{X}$ has nonconstant holomorphic functions.
2) E is a flat bundle, $E \cong \mathcal{O}(\tau)$.

Proof. Lemma 2.4 states that there is a nontrivial π_1-globally generated coherent subsheaf $\mathcal{F}$ of E such that α is contained in the image of $H^1(X, \mathcal{F})$ in $H^1(X, E)$. If $\text{rk}\mathcal{F} < \text{rk}E$ then the absolute stability of E would imply that the line bundle $det\mathcal{F} \in -N_{eff}(X)^+$. Hence $\mathcal{F}$ is not a flat vector bundle. It follows from lemma 2.5 that $\tilde{X}$ must have nonconstant holomorphic functions.

If $\text{rk}\mathcal{F} = \text{rk}E$ then E is generically π_1-globally generated. If $\tilde{X}$ has no nonconstant holomorphic functions, then lemma 2.5 gives that E is a flat vector bundle $\tilde{X} \times_\tau \mathbb{C}^r$, for some representation $\tau : \pi_1(X) \to GL(r, \mathbb{C})$. □

2.2 Flat bundles and holomorphic functions.

It follows from "classical" Hodge theory that if the fundamental group of a Kahler manifold X has an infinite representation into $\mathbb{C}^* = GL(1, \mathbb{C})$ then $H^0(\tilde{X}, \mathcal{O}) \neq \mathbb{C}$. This holds since the Hodge decomposition of $H^1(X, \mathbb{C})$ ($\neq 0$ because of the condition on $\pi_1(X)$) implies that $H^0(X, \Omega^1_X) \neq 0$. The nonconstant functions appear as the integrals on $\tilde{X}$ of the pullback of the nontrivial holomorphic (1,0)-forms in $H^0(X, \Omega^1_X)$. To take similar conclusions from an arbitrary infinite linear representation $\rho : \pi_1(X) \to GL(n, \mathbb{C})$ one needs to use the "recent" non-abelian Hodge theory developed by Simpson [Si88]. This was done by Katzarkov, Pantev, Ramanchandran and Eyssidieux and the result is:

Theorem 2.8. *If a smooth Kahler manifold X has an infinite linear representation of the fundamental group then its universal cover has nonconstant holomorphic functions (see [Ey04] and [Ka97] for projective surfaces).*

Remark: The previous theorem states that there are properties of the fundamental group of a Kahler manifold X that guarantee the existence of nonconstant holomorphic functions on the universal cover $\tilde{X}$. Can we have a similar result when the Kahler condition is dropped? The answer is no (which motivates the search of alternative approaches to the existence of holomorphic function on universal covers). The negative answer follows from the results of Taubes on anti-self-dual structures on real 4-manifolds [Ta92]. Taubes showed that every finitely presented group is the fundamental group of a compact complex 3-fold X that has a a covering family of smooth $\mathbb{P}^1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. This in turn implies that the universal cover $\tilde{X}$ has no non-constant holomorphic functions. The universal cover $\tilde{X}$ also has a covering family of smooth $\mathbb{P}^1$'s with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Any one of these $\mathbb{P}^1$ has a 2-concave and hence pseudoconcave neighborhood since their normal bundle is Griffiths-positive [Sc73]. The conclusion follows since a complex manifold with a pseudoconcave open subset has only constant holomorphic functions. Moreover the variety $\tilde{X}$ with X being a twistor space for a sufficiently generic anti-self-dual metric on the underlying 4-dimensional variety has no meromorpihic functions. Indeed the field of meromorphic functions on $\tilde{X}$ is always a subfield of the field of meromorphic functions in the normal neighborhood of $\mathbb{P}^1$ and the latter is always a subfield of $C(x, y)$ and consists of constant functions only for a generic neighborhood of $\mathbb{P}^1$ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

3. The pullback map for vector bundles and holomorphic functions on universal covers

The presence of nonconstant holomorphic functions on the universal cover $\tilde{X}$ of a Kahler manifold X is many times related with existence of nonisomorphic vector bundles on X which become isomorphic after the pullback to $\widetilde{X}$. For example, the theorem of L.Katzarkov [Ka95] on the Shafarevich conjecture establishes the holomorphic convexity of $\widetilde{X}$ for a projective surface X if $\pi_1(X)$ admits almost faithful linear representations (see also [Ey04]). In this case all the bundles on X corresponding to representations of the same rank of the fundamental group are becoming equal on $\widetilde{X}$. Let X be a projective manifold, $\rho : \tilde{X} \to X$ its universal covering and $\rho^* : Vect(X) \to Vect(\tilde{X})$ the pullback map for isomorphism classes of holomorphic vector bundles. This section investigates the relation between the properties of the pullback map ρ^* and the existence of holomorphic functions on $\tilde{X}$.

We start by considering the case of projective manifolds whose pullback map ρ^* identifies the isomorphism classes that are isomorphic as topological vector bundles. If the pullback satisfy this property, then there are plenty of distinct vector bundles on X whose pullbacks are identified. In particular, any two bundles which can be connected by an analytic deformation are bound to be identified on $\tilde{X}$. This very rich collection of bundles that are identified via the pullback map imply the following result on the algebra of global holomorphic functions on $\tilde{X}$ holds:

Observation 3.1. *Let X be a projective manifold whose pullback map ρ^* identifies isomorphism classes of holomorphic vector bundles that are in the same topological isomorphism class. Then the universal cover $\tilde{X}$ is Stein.*

Proof. Let X be a subvariety of $\mathbb{P}^n$. Consider the the extension of $\Omega^1_{\mathbb{P}^n}|_X$ coming from the Euler exact sequence restricted to X:

$$0 \to \Omega^1_{\mathbb{P}^n}|_X \to \bigoplus^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \tag{3.1}$$

Associated with (3.1) we have the affine bundle $p : A \to X$ over X that consists of the pre-image of the section $1 \in H^0(X, \mathcal{O}_X)$ in the total space $t(\bigoplus^{n+1} \mathcal{O}_X(-1))$. The affine bundle $A \subset t(\bigoplus^{n+1} \mathcal{O}_X(-1))$ is a Stein manifold since the vector bundle $\bigoplus^{n+1} \mathcal{O}_X(-1)$ is negative and A does not intersect the zero section of $\bigoplus^{n+1} \mathcal{O}_X(-1)$.

Let $\tilde{p} : \tilde{A} \to \tilde{X}$ be affine bundle associated with the pullback of (3.1) to $\tilde{X}$ by $\rho : \tilde{X} \to X$. The affine bundle $\tilde{A}$ is an unramified covering of A an hence it is also a Stein manifold. The result follows if it is shown that $\tilde{A}$ is also a vector bundle since then $\tilde{X}$ embedds in $\tilde{A}$ as the zero section and hence is also Stein.

It follows from (3.1) that topologically $\Omega^1_{\mathbb{P}^n}|_X \oplus \mathcal{O} \simeq \bigoplus^{n+1} \mathcal{O}_X(-1)$. Hence by the hypothesis $\rho^*(\bigoplus^{n+1} \mathcal{O}_X(-1))$ is isomorphic, as a holomorphic vector bundle, to the pullback $\rho^*(\Omega^1_{\mathbb{P}^n}|_X \oplus \mathcal{O}_X)$. Hence $\tilde{A}$ is biholomorphic to $t(\rho^*(\Omega^1_{\mathbb{P}^n}|_X))$ and the observation is proved.

□

This section is mainly concerned with the implications of the absence of nonconstant holomorphic functions on $\tilde{X}$ on the pullback map ρ^*. The condition that $\tilde{X}$ has no nonconstant holomorphic functions lies on the opposite side of the conclusion of observation 3.1, stating that $\tilde{X}$ is Stein. We will show that this condition on $\tilde{X}$ has implications that are quite opposite to the assumption of the observation 3.1. More precisely, the absence of nonconstant holomorphic function on $\tilde{X}$ implies that the pullback map ρ^* is almost an imbedding.

The authors believe that the pullback map ρ^* being almost an imbedding should not hold for projective varieties with infinite $\pi_1(X)$; see the remarks at the end of this section. If the authors were correct, this approach would be instrumental in showing that there are always nonconstant holomorphic functions on noncompact universal covers $\tilde{X}$ of projective varieties.

3.1 Pullback map for line bundles.

We describe the implications of the absence of nonconstant holomorphic functions on the universal cover $\tilde{X}$ on the pullback map for line bundles $\rho^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$.

Definition 3.2. *The cone of $\mathbb{Q}$-divisors on X generated by the divisors which become effective on $\tilde{X}$ is denoted by $\tilde{N}_{eff}(X) \subset N^1_{\mathbb{Q}}(X)$ ($\tilde{N}_{eff}(X)^+ = \tilde{N}_{eff}(X) \setminus 0$).*

Suppose that $\tilde{X}$ has no non-constant holomorphic functions. Then the cone $\tilde{N}_{eff}(X)$, which contains $N_{eff}(X)$, does not contain any elements from $-N_{eff}(X)^+$. Assume $-N_{eff}(X)^+ \cap \tilde{N}_{eff}(X) \neq \emptyset$ then there is an divisor effective D of X such that both line bundles $\rho^*\mathcal{O}(D)$ and $\rho^*\mathcal{O}(-D)$ have nontrivial sections. The pairing of these sections gives a non-constant holomorphic function. The previous discussion implies that if $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$ then the image of $\rho^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ is nontrivial and there are the following possibilities:

P1. The cone $\tilde{N}_{eff}(X)^+$ is separated by a hyperplane H from $-N_{eff}(X)^+$.

P1'. The cone $\tilde{N}_{eff}(X)$ coincides with $N_{eff}(X)$. This is a special case of P1. In particular, it holds if $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ and $\mathrm{Pic}(X) = \mathbb{Z}$.

P2. The closure of the cone $\tilde{N}_{eff}(X)$ in $N^1_{\mathbb{R}}(X)$ intersects with the closure of $-N_{eff}(X)$ outside of 0.

The following result describes the kernel of $\rho^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\tilde{X})$ for Kahler manifolds whose universal cover has no nonconstant holomorphic functions.

Proposition 3.3. *Let X be a Kahler manifold such that $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$. Then the kernel of the pullback map $\rho^* : PicX \to Pic\tilde{X}$ is finite and its elements correspond to flat bundles associated with finite characters.*

Proof. Let L be a line bundle in the kernel of ρ^* and let $i : \rho^* L \to \mathcal{O}_{\tilde{X}}$ be an isomorphism with the trivial line bundle. The isomorphism i is not equivariant with respect to the natural $\pi_1(X)$-actions on $\mathcal{O}_{\tilde{X}}$ and on $\rho^* L$ giving respectively $\mathcal{O}_X$ and L on X. Hence there is a $\gamma \in \pi_1(X)$ such that the map $\gamma i \gamma^{-1} i^{-1} \neq \mathrm{Id} : \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$. If $\gamma i \gamma^{-1} i^{-1} \neq \mathrm{c}(\gamma)\mathrm{Id}$ for some constant $c(\gamma)$ then the map $\gamma i \gamma^{-1} i^{-1} : \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}$ would be associated with a nonconstant holomorphic function on $\tilde{X}$, which can not happen.

Therefore, we have an association of elements of $\pi_1(X)$ with nonzero constants, $\gamma \to c(\gamma)$. This association defines a representation $\pi_1(X) \to \mathbb{C}^*$ and this representation has to be finite, since $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ implies that $H^1(X, \mathbb{C})$ must vanish. The line bundles on the kernel of ρ^* are uniquely determined by the representations described above. Thus, $\mathrm{Ker}(\rho^*)$ is dual to $\pi_1(X)^{ab}$ which is a finite group. $\square$

3.2 Pullback map for vector bundles.

The thesis of this paper is that the non-existence of holomorphic functions on $\tilde{X}$ implies that few bundles on X are identified by pulling back to $\tilde{X}$. It is clear that two vector bundles $F \otimes \mathcal{O}(\tau)$ and $F \otimes \mathcal{O}(\tau')$ become isomorphic on $\tilde{X}$ for any bundle F if the rank of the representations τ and τ' is the same. The main result of this section states that if $\tilde{X}$ has no holomorphic functions and two bundles E and E' on X have isomorphic pullback on $\tilde{X}$ then essentially $E = F \otimes \mathcal{O}(\tau)$ and $E' = F \otimes \mathcal{O}(\tau')$ for some bundle F. Our results are mostly for the spaces of absolutely stable bundles but they have a generalization for the spaces of H-stable bundles, if extra conditions on $\tilde{N}_{eff}(X)$ are added.

In order to better understand the pullback map $\rho^* : \mathrm{Vect}(X) \to \mathrm{Vect}(\tilde{X})$ and, in particular, to study its local properties, one should put a structure of an analytic scheme into the sets $\mathrm{Vect}(X)$ and $\mathrm{Vect}(\tilde{X})$. The scheme structure for $\mathrm{Vect}(X)$ for X projective is well understood. Below, we recall the key facts that are relevant to our goals. The analytic scheme structure theory for $\mathrm{Vect}(\tilde{X})$ is less understood. We would like to note that it is not our interest to develop such a theory in this paper. The only facts that we use from $\mathrm{Vect}(\tilde{X})$ are that distinct points correspond to non-isomorphic bundles and that the formal tangent space at a vector bundle E on $\tilde{X}$ exists and it is equal to $H^1(\tilde{X}, EndE)$.

In order to describe the local behavior of the pullback map, it necessary to recall some facts from the theory of deformations of a given vector bundle E on X. The deformation of a vector bundle E over an arbitrary variety splits into the deformation of the projective bundle $\mathbb{P}(E)$ plus a deformation of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ over $\mathbb{P}(E)$. The deformations of $\mathbb{P}(E)$ in the case of a smooth X are parameterized by an analytic subset $B_E \subset H^1(X, EndE), 0 \in B_E$ with the action of the group of relative analytic automorphisms $Aut(E)$ of the bundle $\mathbb{P}(E)$ on B_E. The latter is induced from the natural linear action $Aut(E)$ on $H^1(X, EndE)$ (with adjoint fiberwise action of $PGL(n)$ on the fiber of the bundle End_E). Thus, non-isomorphic bundles (with respect to identical automorphism on X) in the local neighborhood of E are parameterized by the orbits of the group $AutE$ with Lie algebra $H^0(X, End_0(E)$ in $H^1(X, EndE)$.

The space $H^1(X, EndE)$ plays a role of the formal tangent space $T_0(B_E)$ at the point $0 \in B_E$. Natural splitting $EndE = End_0E \oplus \mathcal{O}$ induces a splitting $H^1(X, EndE) = H^1(X, End_0E) \oplus H^1(X, \mathcal{O})$. The local deformation scheme of E maps onto a local deformation scheme of $\mathbb{P}(E)$ with a fiber which is locally isomorphic to $H^1(X, \mathcal{O})$. $H^1(X, \mathcal{O})$ parameterizes the (non-obstructed) deformation scheme of line bundles $\mathcal{O}_{\mathbb{P}(E)}(1)$ in $\mathrm{Pic}(\mathbb{P}(E))$ over the deformation scheme of $\mathbb{P}(E)$ which is generically obstructed.

Let $p : \mathcal{E} \to \Delta$ be an analytic family, over the disc Δ, of vector bundles on X with $E_t = p^{-1}(t)$ as its members. The family E_t gives a deformation of $E = E_0$ and has associated with it a 1st-order deformation cocycle $s \in H^1(X, EndE)$.

Lemma 3.4. *Let $p : \mathcal{E} \to \Delta$ be family of vector bundles on X that is nontrivial at*

$t = 0$. *If the pullback family* $\tilde{p} : \rho^*\mathcal{E} \to \Delta$ *is locally trivial then the kernel of* $\rho^* : H^1(X, EndE) \to H^1(\tilde{X}, End\rho^*E)$ *is nontrivial.*

Proof. The 1st-order deformation cocycle $s \in H^1(X, EndE)$ associated with the family E_t is nontrivial since the family $E_t = p^{-1}(t)$ is nontrivial at $t = 0$. The nontrivial cocycle s is in the kernel of ρ^* since ρ^*s is the 1st-order deformation cocycle associated with the locally trivial family $\tilde{p} : \rho^*\mathcal{E} \to \Delta$ which is trivial. □

Lemma 3.5. *Let* $p : X' \to X$ *be an unramified Galois covering of a smooth projective manifold* X *and* E *a vector bundle on* X. *Then* $H^0(\tilde{X}, End_0p^*E) \neq 0$ *if one of the following holds:*

1) The kernel of $p^* : H^1(X, End_0E) \to H^1(X', End_0p^*E)$ *is nontrivial.*

2) $H^0(X', \mathcal{O}_{X'}) = \mathbb{C}$ *and there is a vector bundle* F *such that* $p^*F = p^*E$ *but* $F \neq E \otimes \mathcal{O}(\chi)$ *for any character* $\chi : \pi_1(X) \to \mathbb{C}^*$.

Proof. Assume that 1) holds then $H^0(\tilde{X}, End_0p^*E) \neq 0$ follows from lemma 2.2 (p must be an infinite unramified covering of X by lemma 2.1).

If 2) holds then there is an isomorphism $i : p^*E \to p^*F$ and $F \not\simeq E \otimes \mathcal{O}(\chi)$ for any character $\chi : G \to \mathbb{C}^*$. Let G be the Galois group of the covering. The isomorphism i is not G-equivariant since otherwise it would descent to an isomorphism $i' : E \to F$ on X. Consider the two possible cases: i) there is a $g \in G$ such that $g^{-1}i^{-1}gi : p^*E \to p^*E$ is a non-scalar endomorphism. Then $g^{-1}i^{-1}gi$ is a nontrivial element in $H^0(X', End_0p^*E)$. ii) For all $g \in G$ the endomorphism $g^{-1}i^{-1}gi$ of p^*E is scalar. Since X' has no nonconstant holomorphic functions, the following holds: $g^{-1}i^{-1}gi = \chi(g)\mathrm{Id}$, $\chi(g) \in \mathbb{C}^*$. Therefore, the map $\chi : G \to \mathbb{C}^*$ defines a character of G and $F = E \otimes \mathcal{O}(\chi)$ which can not happen, since it contradicts the assumption. □

Let $\rho^* : Mod_0(X) \to Vect(\tilde{X})$ be the pullback map, where $Mod_0(X)$ is the moduli space of absolutely stable vector bundles on X. We denote points in $Mod_0(X)$ by the same letters as the corresponding vector bundles. We will say that ρ^* is a *formal local embedding* at $E \in Mod_0(X)$ if the tangent map $\rho^* : H^1(X, EndE) \to H^1(\tilde{X}, EndE)$ is injective.

We start by establishing the main result for absolute stable vector bundles E whose pullback remains simple, i.e. $H^0(\tilde{X}, End_0\rho^*E) = 0$.

Definition 3.6. *A vector bundle* E *over a projective variety* X *is said to be* π_1*-simple if* $H^0(\tilde{X}, End_0\rho^*E) = 0$, *where* $\rho : \tilde{X} \to X$ *is the universal covering of* X.

We show that for π_1-simple vector bundles E there are no non-trivial families of vector bundles F_t on X whose pullbacks ρ^*F_t are isomorphic to ρ^*E.

Theorem 3.7. *Let* X *be a projective manifold such that its universal cover* $\tilde{X}$ *has no nonconstant holomorphic functions. If* E *is an absolutely stable vector bundle on* X *satisfying* $H^0(\tilde{X}, End_0\rho^*E) = 0$, *then:*

a) The pullback map $\rho_0^* : Mod_0(X) \to Vect(\tilde{X})$ is a formal local embedding at E.

b) For any absolutely stable bundle E there are only finite number of bundles F with $\rho^* E = \rho^* F$ and $E = F \otimes \mathcal{O}(\chi)$ with χ a character of $\pi_1(X)$.

Proof. To prove part a) it is enough to show that the tangent map to ρ_0^* at E, $\rho^* : H^1(X, EndE) \to H^1(\tilde{X}, End\rho^* E)$ is injective. The condition $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$ implies $H^1(X, \mathcal{O}_X) = 0$ hence $H^1(X, EndE) = H^1(X, End_0 E)$ The injectivity of $\rho^* : H^1(X, EndE) \to H^1(\tilde{X}, End\rho^* E)$ follows from $H^0(\tilde{X}, End_0 \rho^* E) = 0$ and lemma 3.5 1).

The item b) follows from 2) of lemma 3.5 and the finiteness of the character group of $\pi_1(X)$. The finiteness of the character group follows from $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$.

□

To establish the thesis of this paper, we need to consider the technically more challenging case where E is an absolute stable which is not π_1-simple. The next theorem establishes that after a finite covering the vector bundle E has a direct sum decomposition whose direct summands are a tensor product of a π_1-simple vector bundle with a flat bundle.

Theorem 3.8. *Let X be a projective manifold such that its universal cover $\tilde{X}$ has no nonconstant holomorphic functions. Let E be an absolute stable vector bundle on X satisfying $H^0(\tilde{X}, End_0 \rho^* E) \neq 0$. Then there is a normal subgroup $\pi_1(X') \subset \pi_1(X)$ corresponding to a finite unramified covering $p : X' \to X$ satisfying:*

i) $p^* E \simeq E'_1 \otimes \mathcal{O}(\tau_1) \oplus ... \oplus E'_m \otimes \mathcal{O}(\tau_m)$, $\tau_i : \pi_1(X') \to GL(k, \mathbb{C})$ with $k \geq 1$.

ii) The vector bundles E'_i are π_1-simple.

iii) The natural action of the finite group $G = \pi_1(X)/\pi_1(X')$ on X' extends to the action on $p^* E$ which permutes subbundles $E'_i \otimes \mathcal{O}(\tau_i)$ and this action gives the imbedding of G into S_m.

iv) Let $G_1 \subset G$ be subgroup which acts identically on $E'_1 \otimes \mathcal{O}(\tau_1) \subset p^* E$ then $E'_1 \otimes \mathcal{O}(\tau_1)$ descends to the bundle $E'_1 \otimes \mathcal{O}(\tau'_1)$ on $X_1 = X'/G_1$ with $p_1 : X_1 \to X$ being a nonramified covering of degree $rkE/(rkE'_1 \otimes \mathcal{O}(\tau_1))$ and $E = p_* E'_1 \otimes \mathcal{O}(\tau'_1)$

Proof. We start the proof by claiming properties of the algebra $A = H^0(\tilde{X}, End\rho^* E)$, that will be settled later. In the next paragraphs, these claims will be used to establish the theorem. We claim that the algebra $A \equiv H^0(\tilde{X}, End\rho^* E) = \bigoplus_{i=1}^m M_k$ where M_k is the algebra of $k \times k$ matrices for a $k < r = \text{rk } E$; the action of $\pi_1(X)$ on the algebra A has no nontrivial $\pi_1(X)$-invariant ideals ($\pi_1(X)$ acts transitively on the m direct summands of $A = \bigoplus_{i=1}^m M_k$). We also claim that the action of the algebra A on $\rho^* E$ is such that each direct summand of A acts on each fiber $(\rho^* E)_x \simeq \mathbb{C}^r$, $x \in \tilde{X}$, as the same multiple lM_k of the standard representation of M_k, $r = lmk$.

The $\pi_1(X)$-action on $A = \bigoplus_{i=1}^m M_k$ permutes the m simple direct summands, M_k, and therefore gives a homomorphism $\sigma : \pi_1(X) \to S_m$. Let $p : X' \to X$ be the finite unramified Galois covering of X associated with the kernel of σ. By construction the direct summands of A are $\pi_1(X')$-invariant. The claims describing the structure of A and the action of A on $\rho^* E$ imply $\rho^* E = \tilde{E}_1 \otimes \mathcal{O}^k \oplus ... \oplus \tilde{E}_m \otimes \mathcal{O}^k$. Moreover, since the direct summands of A are $\pi_1(X')$-invariant the summands $\tilde{E}_i = {\rho'}^* E'_i$ where E'_i are vector bundles of rank l on X' and $\rho' : \tilde{X} \to X'$ is the universal cover of X'.

On X' the bundle $p^* E$ decomposes into $p^* E = E'_1 \otimes \mathcal{O}(\tau_1) \oplus ... \oplus E'_m \otimes \mathcal{O}(\tau_m)$ ($\tau_i : \pi_1(X') \to GL(k, \mathbb{C})$) giving i). The claim that there are no $\pi_1(X)$-invariant ideals of A implies that the group $G = \pi_1(X)/\pi_1(X')$ acts on $p^* E$ permuting transitively the direct summands, thus proving iii). Part ii) follows from $H^0(\tilde{X}, End({\rho'}^* E'_i \otimes \mathcal{O}^k)) = M_k$. If $H^0(\tilde{X}, End_0 {\rho'}^* E'_i) \neq 0$ then group of global sections $H^0(\tilde{X}, End({\rho'}^* E'_i \otimes \mathcal{O}^k))$ would be larger then M_k contradicting the claim about A.

Let $G_1 \subset G$ be the subgroup which stabilizes the direct summand $E'_1 \otimes \mathcal{O}(\tau_1)$ of $p^* E$. Let $p_1 : X_1 \to X$ be the intermediate covering where X_1 is the quotient X'/G_1. Then the vector bundle $E'_1 \otimes \mathcal{O}(\tau_1)$ descends to X_1. Moreover, the bundle $p^* E$ descends to X_1 and it decomposes into a direct sum $E'_1 \otimes \mathcal{O}(\tau_1) \oplus E_N$.

We want to show that $p_{1*}(E'_1 \otimes \mathcal{O}(\tau_1)) = E$. Consider the direct images $p_{1*}(E'_1 \otimes \mathcal{O}(\tau_1))$ and $p_{1*} p_1^* E$ on X. The bundle $p_{1*} p_1^* E$ has a natural decomposition into $E \oplus E_c$. The natural projection $i_1^* : p_1^* E \to E'_1 \otimes \mathcal{O}(\tau_1)$, which is identity on $E'_1 \otimes \mathcal{O}(\tau_1)$, induces a map $r : p_{1*} p_1^* E \to p_{1*}(E'_1 \otimes \mathcal{O}(\tau_1))$. Also denote by r the restriction of r on the direct summand $E \subset p_{1*} p_1^* E$. We claim that r is an isomorphism. It follows from the fiberwise description of r. Let $\mathbb{C}_x^{mlk} = \Sigma_{i=1}^m \mathbb{C}_{i,x}^{lk}$ be the direct sum decomposition of the fiber of $p^* E$ at $x \in X'$ corresponding to the decomposition into the direct summands $E'_i \otimes \mathcal{O}(\tau_i)$. Let $g_i \in G$ be the representatives of cosets G/G_1. Then for a $x' \in X$ its pre-image $p_1^{-1} x' \subset X_1$ is equal to $\bigcup g_i x_1$ and the fiber of $E'_1 \otimes \mathcal{O}(\tau_1)$ over $g_i x_1$ is naturally isomorphic to $\mathbb{C}_{i,x}^{lk}$. The map r becomes the trace map for the action of G on $p^* E$ which implies that r is a fiberwise isomorphism. This proves iv).

Claim: $A = H^0(\tilde{X}, End \rho^* E)$ is a subalgebra of the matrix algebra M_r, $r = \mathrm{rk} E$.

First, we prove that A is a finite dimensional algebra. Let $\mathcal{A}$ be the subsheaf of $End \rho^* E$ generated by the global sections of $End \rho^* E$. The sheaf $End \rho^* E$ is a sheaf of matrix algebras and $\mathcal{A}$ is a sheaf of subalgebras since we can add and multiply sections. The sheaf $\mathcal{A}$ is invariant under the action of $\pi_1(X)$ and defines a coherent subsheaf $\mathcal{A}' \subset End E$ on X with $\mathcal{A} = \rho^* \mathcal{A}'$. The absolute stability of E implies that $det \mathcal{A}' \in -N_{eff}(X)$ by i) of corollary 1.6. The lemma 2.5 implies that $\mathcal{A}'$ is isomorphic to the sheaf of sections of a flat vector bundle since $(det \mathcal{A}')^{-k}$ ($det \mathcal{A}' \in -N_{eff}(X)$) has a nontrivial section for some k and $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$. Hence $\mathcal{A} = \rho^* \mathcal{A}' \simeq \mathcal{O}_{\tilde{X}}^q$. It follows from $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$, that the algebra A is finite dimensional.

We want to show that the algebra A is isomorphic to $\mathcal{A} \otimes k(x) \subset End \rho^* E \otimes k(x) \cong M_r$ for all x in $\tilde{X}$, where $k(x)$ is the residue field at x and $r = \mathrm{rk} E$. Consider the exact sequence $0 \to \mathcal{A} \otimes \mathcal{I}(x) \to \mathcal{A} \to \mathcal{A} \otimes k(x) \to 0$, $\mathcal{I}(x)$ the ideal sheaf of the point x. Since $\mathcal{A}$ is generated by its global sections it follows that the morphism $A = H^0(\tilde{X}, \mathcal{A}) \to H^0(X, \mathcal{A} \otimes k(x)) = \mathcal{A} \otimes k(x)$ is a surjection. If the morphism is also an injection we get the desired isomorphism $A \cong \mathcal{A} \otimes k(x)$. The injectivity follows from $H^0(\tilde{X}, \mathcal{A} \otimes \mathcal{I}(x)) = 0$, which holds since any nontrivial section s of $End \rho^* E$ is nowhere vanishing (see the end of previous paragraph).

Claim: The algebra A is semisimple.

The semisimplicity of A is equivalent to the maximal nilpotent ideal I_m of A being the zero ideal. The algebra A comes with a natural $\pi_1(X)$-action. The maximal nilpotent ideal is a $\pi_1(X)$-invariant ideal of A. Every nontrivial $\pi_1(X)$-invariant ideal I of A defines naturally a nontrivial subsheaf $\mathcal{I}' \subset \mathcal{A}' \subset EndE$.

Suppose that the ideal I_m is nontrivial. The nilpotent condition, $I_m^k = 0$ for some k, implies that the subsheaf $\mathcal{I}'_m E \subset E$ satisfies $\mathrm{rk}(\mathcal{I}'_m E) < \mathrm{rk} E$. Under these conditions we have an the exact sequence:

$$0 \to \mathcal{K} \to \mathcal{I}'_m \otimes E \to \mathcal{I}'_m E \to 0$$

with $\mathcal{K} \neq 0$. We will show that the subsheaf $\mathcal{K}$ is an H-destabilizing subsheaf of $\mathcal{I}'_m \otimes E$ for any polarization H of X. This gives a contradiction, since $\mathcal{I}'_m \otimes E$ is H-semistable. The sheaf $\mathcal{I}'_m \otimes E$ is the tensor product of the two H-semistable sheaves E and $\mathcal{I}'_m$ ($\mathcal{I}'_m$ is a flat bundle) (the tensor product of two H-semistable sheaves is H-semistable).

We need to get the destabilizing inequality $[\mathrm{rk}(\mathcal{I}'_m \otimes E) det\mathcal{K} - \mathrm{rk}\mathcal{K} det(\mathcal{I}'_m \otimes E)].H^{n-1} > 0$. Using $det\mathcal{K} = \mathrm{rk}\mathcal{I}'_m detE - det\mathcal{I}'E$ and $\mathrm{rk}E det\mathcal{I}'_m E - \mathrm{rk}\mathcal{I}'_m E detE \in N_{eff}(X)^+$ (E is an absolute stable bundle), it follows that $\mathrm{rk}(\mathcal{I}'_m \otimes E) det\mathcal{K} - \mathrm{rk}\mathcal{K} det(\mathcal{I}'_m \otimes E) \in -N_{eff}(X)^+$. Hence we obtain the desired contradiction, which implies that the $\pi_1(X)$-invariant ideal I_m must be the zero ideal and A is semisimple.

Claim: A has no proper $\pi_1(X)$-invariant ideals.

We proved that A is semisimple and hence $A = \Sigma_{i=1}^m M_{r_i} \subset M_r$ with $r_1 + ... + r_m \leq r$. Recall that if I is an ideal of $A = \Sigma_{i=1}^m M_{r_i}$ (A acts on $\mathbb{C}^r$) such that $I\mathbb{C}^r = \mathbb{C}^r$ then $I = A$. Let I be a $\pi_1(X)$-invariant ideal of A and $\mathcal{I}$ the associated subsheaf of $\mathcal{A}$. There are two cases: i) $\mathrm{rk}\mathcal{I}\rho^* E = r$ and ii) $\mathrm{rk}\mathcal{I}\rho^* E < r$.

If i) holds, then $\mathcal{I} \otimes k(x) \subset \mathcal{A} \otimes k(x) \cong \Sigma_{i=1}^m M_{r_i}$ is such that $\mathcal{I} \otimes k(x)(\rho^* E)_x = (\rho^* E)_x$. Hence by what was recall above it follows $\mathcal{I} \otimes k(x) = \mathcal{A} \otimes k(x) \cong \Sigma_{i=1}^m M_{r_i}$ or equivalently $I = A$. If ii) holds then the subsheaf $\mathcal{I}'E \subset E$ satisfies $\mathrm{rk}\mathcal{I}'E < \mathrm{rk}E$. The same argument used in the claim above can be applied to show $I = 0$ settling the claim.

Claim: The algebra A is equal to mM_k and the action of A on any fiber $(\rho^* E)_x$ induces for each M_k a representation equal to a multiple of a standard rank k representation of M_k.

We already showed that $A = \Sigma_{i=1}^m M_{r_i}$ with $r_1 + ... + r_m \leq r$. Since each M_{r_i} is simple it follows that the action of $\pi_1(X)$ preserves the ideals of A corresponding to the sums of the M_{r_i}. Since any $\pi_1(X)$-invariant ideal I of A is either trivial or the full A, it follows that the $\pi_1(X)$-action acts transitively on the blocks M_{r_i}. This in particular implies that all the r_i must be equal to the same k.

Finally, we show that the representation of each $A = M_k$ in M_r is a multiple of the standard representation. Any irreducible representation of M_k is the standard representation or the zero representation. The presence of a zero representation as an irreducible component of the representation of M_k in M_r would imply that $\mathcal{A}'E \neq E$ which is not possible from the discussion above.

□

The following lemma follows from our results and theorem 2.8.

Lemma 3.9. *Let X be a Kahler manifold such that $\tilde{X}$ has no nonconstant holomorphic functions. Then for any linear representation τ of $\pi_1(X)$ the vanishings $H^1(X, \mathbb{C}(\tau)) = H^1(X, \mathcal{O}(\tau)) = 0$ must hold.*

Proof. The hypothesis $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$ and theorem 2.8 imply that the representation τ must be finite. Let $f : X' \to X$ of X be the finite covering corresponding to the kernel $G \subset \pi_1(X)$ of the representation τ. The manifold X' is a compact Kahler manifold with the same universal cover as X hence $H^1(X', \mathbb{C}^k) = H^1(X', \mathcal{O}^k) = 0$ hold since $H^0(\tilde{X}, \mathcal{O}) = \mathbb{C}$. The result follows from lemma 2.1 which states that the maps $f^* : H^1(X, \mathbb{C}(\tau)) \to H^1(X', \mathbb{C}^k)$ and $f^* : H^1(X, \mathcal{O}(\tau)) \to H^1(X', \mathcal{O}^k)$ are embeddings. □

Theorem 3.10. *Let X be a projective manifold whose universal cover has only constant holomorphic functions. Then:*

a) The pullback map $\rho_0^* : Mod_0(X) \to Vect(\tilde{X})$ is a local embedding.

b) For any absolutely stable bundle E there are only finite number of bundles F with $\rho^*E = \rho^*F$.

c) The vector bundle E determines a finite unramified cover $p : X' \to X$ of degree $d \leq rkE!$. On X' there is a fixed collection of π_1-simple vector bundles $\{E_i'\}_{i=1,...,m}$ such that a vector bundle F on X satisfies $\rho^*F \simeq \rho^*E$ if and only if:

$$p^*F = E_1' \otimes \mathcal{O}(\tau_1) \oplus ... \oplus E_1' \otimes \mathcal{O}(\tau_m)$$

The bundles $\mathcal{O}(\tau_i)$ are flat bundles associated with finite linear representations of $\pi_1(X')$ of a fixed rank k with $rkE|k$.

Proof. In theorem 3.7 we dealt with the case of vector bundles E such that $H^0(\tilde{X}, End_0\rho^*E) = 0$. We proceed to consider the case $H^0(\tilde{X}, End_0\rho^*E) \neq 0$.

a) Let $Mod_0(X, V)$ be the moduli space of absolutely stable bundles with the same Chern classes as V. The formal tangent space of $Mod_0(X, V)$ at V is given by $H^1(X, EndV)$. The vector bundle $EndV$ is semistable with $detEndV = \mathcal{O}$ and is the direct sum

$EndE = \bigoplus_{i=1}^{l} F_i$ of absolutely stable bundles with $\mu_H(F_i) = 0$ by corollary 1.8. In fact, $(detF_i)^l = \mathcal{O}_X$ for some l since $Pic^0(X)$ is torsion as $H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \mathbb{C}$.

The kernel of the tangent map $\rho^* : H^1(X, EndV) \to H^1(\tilde{X}, End\rho^* V)$ is the direct sum of the kernels of $\rho_i^* : H^1(X, F_i) \to H^1(\tilde{X}, \rho^* F_i)$. Assume that one of the maps ρ_i^* is not injective. Proposition 2.7 implies that if $ker\rho_i^* \neq 0$ then $F_i = \mathcal{O}(\tau)$. This leads to a contradiction since the previous lemma 3.9 states that $H^1(X, F_i) = 0$ for the flat bundles F_i. This concludes the proof that ρ^* is a local imbedding.

Part b) is a consequence of c), hence we first consider c). Theorem 3.8 states that $\rho^* E$ has the decomposition $\rho^* E \cong \tilde{E}_1 \otimes \mathcal{O}^k \oplus ... \oplus \tilde{E}_m \otimes \mathcal{O}^k$ with simple vector bundles $\tilde{E}_i$. If $\rho^* F \cong \rho^* E$ then $\rho^* F$ inherits also a decomposition $\rho^* F \cong \tilde{F}_1 \otimes \mathcal{O}^k \oplus ... \oplus \tilde{F}_m \otimes \mathcal{O}^k$ with $\tilde{F}_i \otimes \mathcal{O}^k \cong \tilde{E}_i \otimes \mathcal{O}^k$. Since the $\tilde{E}_i$ are simple vector bundles on $\tilde{X}$ it follows that $\tilde{F}_i \cong \tilde{E}_i$. Also by theorem 3.8, we have a finite covering $p : X' \to X$ where $p^* E$ decomposes into:

$$p^* E \simeq E_1' \otimes \mathcal{O}(\tau_1') \oplus ... \oplus E_m' \otimes \mathcal{O}(\tau_m')$$

The vector bundles E_i' are π_1-simple. Equally, it follows from theorem 3.8 that $p^* F$ also decomposes into:

$$p^* F \cong F_1' \otimes \mathcal{O}(\tau_1'') \oplus ... \oplus F_m' \otimes \mathcal{O}(\tau_m'') \tag{3.2}$$

with $\rho'^* F_i' = \tilde{F}_i$. It follows from lemma 3.5 that $E_i' \otimes \mathcal{O}(\chi) \cong F_i'$ for some character $\chi : \pi_1(X') \to \mathbb{C}^*$, since $\rho'^* E_i' = \tilde{E}_i \cong \tilde{F}_i = \rho'^* F_i'$ and the E_i' are π_1-simple. Hence c) follows from the decomposition (3.2).

To prove b) we first claim that there is a finite unramified Galois covering $\hat{p} : \hat{X} \to X$ associated with E such that $\rho^* F \cong \rho^* E$ if and only if $\hat{p}^* F \cong \hat{p}^* E$. Theorem 3.8 and the proof of c) give that there is a covering $p : X' \to X$ such that if a bundle F on X satisfies $\rho^* F \cong \rho^* E$ then:

$$p^* F \cong E_1' \otimes \mathcal{O}(\tau_1) \oplus ... \oplus E_m' \otimes \mathcal{O}(\tau_m) \tag{3.3}$$

where the $\mathcal{O}(\tau_i)$ are all flat bundles of equal rank k. The variety $M(\pi_1(X'), k)$ of the representations of $\pi_1(X')$ into $GL(k, \mathbb{C})$ is a finite set of points. This follows from $M(\pi_1(X'), k)$ being zero dimensional which holds since the tangent space at each representation $\tau : \pi_1(X') \to GL(k, \mathbb{C})$ is $H^1(X', End\mathcal{O}(\tau)) = 0$ (lemma 3.9). The finiteness of the set of representations implies the existence of a finite Galois cover $\hat{p} : \hat{X} \to X$ where $\hat{p}^* F \cong \hat{p}^* E$ if $\rho^* F \cong \rho^* E$. The result follows then by the lemma:

Lemma 3.11. *Let $f : Y \to X$ be a finite unramified Galois covering of X and E an absolutely stable bundle on X. If F is a vector bundle on X such that $f^* F \cong f^* E$ then F belongs to a finite collection of isomorphism classes of vector bundles on X.*

Proof. If $f^* E$ is a simple vector bundle then the part 2) of lemma 3.5 gives the result. More precisely, it shows that $F \cong E \otimes \mathcal{O}(\chi)$ where $\chi : G \to \mathbb{C}^*$ is a character of the Galois group G of the cover f.

If f^*E is not simple applying the argument in theorem 3.8 we get that $f^*E \cong E_1 \otimes \mathcal{O}_Y^k \oplus ... \oplus E_m\mathcal{O}_Y^k$ and $H^0(Y, Endf^*E) = \bigoplus_{i=1}^m M(k)$ for some k dividing rk$E$. The vector bundles E and F are quotients of two different actions of the Galois group G on f^*E. The action of G on F^*E giving F is determined by the action giving E and a representation $\tau : G \to GL(mk, \mathbb{C})$. Our result follows since the number of isomorphism classes of representations $\tau : G \to GL(mk, \mathbb{C})$ is finite. □

□

Remark: We have a similar result for H-stable bundles if $\tilde{N}_{eff}(X)$ satisfies $P1$ or $P1'$ on section 3.1.

What about the map of the space of all bundles (omitting the discussion of whether it can be well defined)? Notice that for any given filtration of saturated subsheaves in a vector bundle V there is a blow up X' of X such that the pullback of this filtration becomes a filtration of vector bundles (see Moishezon [Mo69] lemma 3.5). In particular, for any vector bundle V on X one can use the Harder-Narasimhan filtration. Since the algebra of holomorphic functions on $\tilde{X}$ does not change after changing blowning up, any conclusion about the function theory for $\tilde{X}'$ holds for $\tilde{X}$. It follows from the above that if $P1'$ holds then the pullback map for all bundles is non-injective modulo representations of $\pi_1(X)$ only if there are cocycles $\alpha \in H^1(X, V)$ such that $\rho^*\alpha = 0$.

The results of this paper can potentially be used to show that the universal cover of a projective variety has a nonconstant holomorphic function. This is explained below.

Proposition 3.12. *Let X be a projective manifold of dimension n and X' be an infinite unramified cover of X then $H^n(X', \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on X'.*

Proof. The result follows from Cech cohomology and Leray coverings if any noncompact cover of a n-dimensional projective variety is covered by n Stein open subsets. Pick $n-1$ generic hyperplane sections H_i and let $C = H_1 \cap ... \cap H_{n-1}$. By Lefschetz theorem C is a smooth curve such that $\pi_1(C) \to \pi_1(X)$ is a surjection. This implies that the pre-image of C in X' is an irreducible noncompact curve C'. Hence C' is Stein (Behnke-Stein theorem). The infinite cover X' is covered by the pre-images U_i of $X \setminus H_i$ in X' and a neighborhood of C'. The pre-images U_i are Stein open subsets of X' since any unramified cover of a Stein manifold is Stein. To conclude, C' has an open Stein neighborhood in X' since C' is a Stein closed subvariety of X' (Siu [Si76]). □

Remark: The proposition 3.12 implies that for surfaces the structure of the space of the moduli space of vector bundles on $\widetilde{X}$ should be similar to the structure of the moduli space of vector bundles on a curve. Namely the groups $H^2(\tilde{X}, \mathcal{F})$ vanish for any coherent sheaf $\mathcal{F}$. In particular, there are no algebraic obstructions in $H^2(\tilde{X}, EndE)$ to deform a vector bundle E along a cocycle in $H^1(\tilde{X}, EndE)$ though there may be an analytic one (problem of convergency). We expect that any bundle of rank ≥ 2 has a complete flag of subbundles if there is a complete flag of topological subbundles. This would imply that the K-group $K_0(\widetilde{X})$ reduces to $\mathrm{Pic}(X)\times\mathbb{Z}$. The above motivates the authors' expectation that many different bundles on X coincide after pulling back to $\widetilde{X}$.

Remark: The last remark may also provides a clue to the proof of $H^0(\tilde{X},\mathcal{O}) \neq \mathbb{C}$ for the universal covers $\tilde{X}$ of projective surfaces X with infinite fundamental group. Indeed let X be a projective surface and E a stable vector bundle on X with additional property that the bracket $[\theta,\theta] \neq 0 \in H^2(X, EndE)$ for sufficiently generic cocylce $\theta \in H^1(X, EndE)$. This provides with an obstruction to the deformation of the bundle E in direction θ. However on the universal covering $\tilde{X}$ the cocycle θ has a trivial bracket since the groups $H^2(X, EndE)$ are zero. Thus ρ^*E should have a deformation along $\rho^*\theta$ with the first germ $\pi_1(X)$-invariant. If we would be able to extend it to a $\pi_1(X)$-equivariant deformation of ρ^*E, we would obtain a deformation of E along θ. This would imply that cocycles with nontrivial brackets have the same image as cocycles with trivial brackets. This allows to construct holomorphic functions on $\tilde{X}$ using our approach. We plan to address this in the next publication.

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