LOCAL STRUCTURE OF CLOSED SYMMETRIC 2-DIFFERENTIALS

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0. INTRODUCTION

In the authors's previous work on symmetric differentials and their connection to the topological properties of the ambient manifold, a class of symmetric differentials was introduced: closed symmetric differentials ([BoDeO11] and [BoDeO13]). Closed symmetric differentials are characterized by the possibility to locally decompose the differential as a product of closed holomorphic 1-differentials in a neighborhood of a point of the manifold. The property of being closed is conjecturally described by a non-linear differential operator (in the case of dimension 2 and degree 2 this differential operator comes from the Gaussian curvature, see section 2.1).

In this article we give a description of the local structure of closed symmetric 2-differentials on complex surfaces, with an emphasis towards the local decompositions as products of 1-differentials. Recall that there is a general obstruction for a symmetric 2-differential to have a decomposition as a product of 1-differentials around a point x in the complex surface X, it might be impossible to order the two foliations defined by w near x (we then say that w is not locally split at x). This obstruction can be removed via a ramified covering of X, hence the results will be given for symmetric differentials that are locally split.

We show that a closed symmetric 2-differential w of rank 2 (i.e. defines two distinct foliations at the general point) has a subvariety $B_w \subset X$ outside of which w is locally the product of closed holomorphic 1-differentials. The main result, theorem 2.6, gives a complete description of a (locally split) closed symmetric 2-differential in a neighborhood of a general point of B_w. A consequence of the main result is that the differential w still has a local decomposition into a product of closed 1-differentials (in a generalized sense) at the points of B_w. The closed 1-differentials involved in the local decompositions

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might have to be multi-valued and acquire singularities along B_w. Note that if we were considering local decompositions of a locally split holomorphic symmetric 2-differential into a product of 1-differentials (not necessarily closed), then the 1-differentials involved can be chosen to be holomorphic, i.e. no singularities need to occur. On the other hand, it is also true that by multiplying one of the 1-differentials by an arbitrary function and the other by its inverse, that arbitrary singularities can occur in the decomposition. An important feature of decompositions of symmetric 2-differentials of rank 2 as products of closed 1-differentials is that they are unique up to multiplicative constants, hence there is no ambiguity on the singularities that occur.

The singularities that occur in the decomposition of a closed holomorphic symmetric 2-differential w when we require that the 1-differentials are closed can be essential singularities along the locus B_w. A key feature of theorem 2.6 giving the local structure of w around points in B_w is that we have a control on these essential singularities, they come from exponentials of meromorphic functions acquiring poles of a bounded order along B_w. Before describing the nature of the bound, we need to describe our result characterizing the locus B_w. In the case w is locally split (always the case after a ramified cover), we show that any irreducible component of B_w must be simultaneously a leaf of both foliations defined by w. The bound on the order of the poles along an irreducible component of B_w is the order of contact of both foliations along that irreducible component.

This article addresses the case of closed symmetric 2-differentials, we expect that a straightforward generalization of our methods will provide similar results on the local structure of closed symmetric differentials of arbitrary degree and give control of the singularities that occur on the decompositions as product of closed 1-differentials.

1. General set up

A symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ on a complex manifold X defines at each point where $w(x) \neq 0$ a cone in tangent space T_xX with vertex the origin and defined at infinity ($\mathbb{P}^{n-1}$) by a variety of degree m. If X is a complex surface then one gets a distribution of d ($d \leq m$) lines, which will be integrable, defining a non-singular d-web at the general point. In higher dimensions the cones will not be necessarily union of hyperplanes and even if they are hyperplanes their distributions need not be integrable. Here, we should note that the class of symmetric differentials that is studied in this work, closed symmetric differential (see below), will in all dimensions be connected to webs on the manifold.

Definition 1.1. *A symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ is split if it has a decomposition:*

$$w = \psi_1 ... \psi_m$$

where the ψ_i are meromorphic 1-forms or equivalently if $w = \mu_1 ... \mu_m$, with $\mu_i \in H^0(X, \Omega^1_X \otimes L_i)$, where L_i are line bundles on X.

Geometrically being split means that the symmetric differential defines hyperplane distributions and moreover they can be numbered consistently globally.

Definition 1.2. *A symmetric differential w on X has rank r if at a general point $x \in X$ $w(x)$ defines r distinct hyperplanes in $T_x X$.*

Definition 1.3. *A symmetric differential w on X is said to have a holomorphic closed decomposition if:*

$$w = \mu_1 ... \mu_m \ , \qquad \mu_i \text{ closed holomorphic 1-forms} \tag{1.1}$$

and a holomorphic closed decomposition at x if x has an analytic neighborhood where (1.1) holds.

Definition 1.4. *A symmetric differential $w \in H^0(X, S^m \Omega^1_X)$ is said to be:*

1) closed, if w has an holomorphic closed decomposition at a general point $x \in X$.

2) of the 1st kind, if w has holomorphic closed decompositions at all $x \in X$.

Remarks: 1) The class of closed symmetric differentials of the 1st kind plays a special role in the motivation for considering closed symmetric differentials as a class of symmetric differentials having a stronger connection to the topology of the ambient manifold. We expand on this point below.

2) Our definitions of closed and 1st kind coincide with the usual definitions when m=1, i.e. holomorphic 1-forms. Our definition of closed asks for a holomorphic 1-form to be locally exact somewhere which by the identity principle implies it is locally exact everywhere and hence closed in the usual sense. Hence, for $m = 1$ our notions of closed and 1st kind coincide.

3) If the degree $m > 1$, then closed no longer implies of the 1st kind. This has far reaching geometric consequences and the main results of this work concern the locus where this failure comes from and the structure of the closed symmetric differentials near this locus.

Definition 1.5. *The locus of X where a closed symmetric differential w fails to be of the 1st kind at, $B_w = \{x \in X |\ w \text{ has no holomorphic closed decomposition at } x\}$, will be called the breakdown locus of w.*

A key feature of holomorphic closed decompositions is that they have rigidity properties. The level of rigidity has to do with a familiar notion in the theory of webs, the abelian rank of a web.

Definition 1.6. *Given the germ $w_x \in S^m\Omega^1_{X,x}$ with the holomorphic closed decomposition*

$$w_x = \mu_1 ... \mu_m \tag{1.2}$$

where $\mu_i \in \Omega^1_{X,cl,x}$ ($\Omega^1_{X,cl}$ is the sheaf of closed holomorphic 1-forms on X), we call an m-tuple $(f_1, ..., f_m) \in \mathcal{M}^m_x$ satisfying:

$$\sum_{i=1}^{m} f_i \mu_i = 0 \quad \text{with} \quad df_i \wedge \mu_i \equiv 0.$$

an abelian relation of the decomposition (1.2). The abelian rank of the decomposition (1.2) is the dimension of the $\mathbb{C}$-vector space consisting of all abelian relations of (1.2). The abelian rank of a closed symmetric differential $w \in H^0(X, S^m\Omega^1_X)$ is the abelian rank of any holomorphic closed decomposition at the general point of x.

Remarks: 1) The definition of abelian rank of a closed symmetric differential w is well defined, since there is an analytic subvariety of $R \subset X$ such that all holomorphic closed decompositions of w_x, $\forall x \in X \setminus R$, have the same abelian rank.

2) It is a classical result of web theory that the abelian rank of a decomposition (1.2) is finite if rank(w)=m, with upper bounds depending on the dimension and degree (for dimension 2 this is a result of G.Bol and also W.Blaschke see for example [ChGr78], [He01] and [He04] for information on webs).

3) A general germ of a closed symmetric differential has trivial abelian rank.

We concentrate our attention to the case of trivial abelian rank which is the generic case and holds trivially for all closed symmetric 2-differentials (and rank 2).

Proposition 1.1. *Let X be a connected manifold and $w \in H^0(X, S^m\Omega^1_X)$ be a closed symmetric differential with an holomorphic closed decomposition $w = \mu_1 ... \mu_m$. If the abelian rank of w is trivial, then all holomorphic closed decomposition $w = \eta_1 ... \eta_m$ of w on X have the closed 1-forms $\eta_i = c_i \mu_i$ with $c_i \in \mathbb{C}^*$ and $\prod_{i=1}^m c_i = 1$.*

Proof. Suppose $w = \eta_1 ... \eta_m$ is an holomorphic closed decomposition of w and assume the η_i are ordered such that $\eta_i \wedge \mu_i = 0$. The condition $\eta_i \wedge \mu_i = 0$ in conjunction with η_i and μ_i being closed implies that $\eta_i = f_i \mu_i$ with $f_i \in \mathcal{M}(X)$ and $df_i \wedge \mu_i = 0$. Moreover $\mu_1 ... \mu_m = \eta_1 ... \eta_m$ gives:

$$\prod_{i=1}^{m} f_i = 1 \tag{1.3}$$

Pick a simply connected open set $U \subset X$ where $f_i|_U \in \mathcal{O}^*(U)$. Taking the logarithm and differentiating (1.3) restricted to U and using the identity principle we obtain:

$$\sum_{i=1}^{m} \frac{df_i}{f_i} = 0 \tag{1.4}$$

but $df_i \wedge \mu_i = 0$, hence $df_i = g_i \mu_i$ with $g_i \in \mathcal{M}(X)$ and $dg_i \wedge \mu_i = 0$. It follows that (1.4) gives rise to the abelian relation at a general point $x \in X$:

$$\sum_{i=1}^{m} (\frac{g_i}{f_i})_x (\mu_i)_x = 0$$

The abelian rank of w being trivial implies that the $(g_i)_x = 0$ and hence $df_i = 0$ on X, i.e. $f_i = c_i \in \mathbb{C}^*$.♦

A consequence of this proposition, see below, is the decomposition of symmetric differentials of the 1st kind with abelian rank 0 into a product of twisted closed holomorphic 1-forms $\phi_i \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_i})$. In [BoDeO13] we use such decompositions to characterize the origins and the geometric implications of symmetric 2-differentials of the 1st kind.

Corollary 1.2. *Let X be a complex manifold and $w \in H^0(X, S^m\Omega^1_X)$ be of the 1st kind with trivial abelian rank. Then there is a finite unramified cover $f : X' \to X$ (unnecessary if w is split) for which f^*w has a decomposition:*

$$f^*w = \phi_1 ... \phi_m$$

where $\phi_i \in H^0(X', \Omega^1_{X',cl} \otimes \mathbb{C}_{\rho_i})$, where the $\mathbb{C}_{\rho_i}$ are local systems of rank 1 on X' such that $\mathbb{C}_{\rho_1} \otimes ... \otimes \mathbb{C}_{\rho_m} \simeq \mathbb{C}$.

Proof. The differential w being of first kind implies that locally w is split, but w might fail to be globally split. This failure is measured by the monodromy coming from the local ordering of the foliations, i.e. we obtain a representation $\sigma : \pi_1(X, x) \to S_m$. Associated to this representation we get an unramified cover $f : X' \to X$ with degree a factor of $m!$ such that f^*w is split.

From now on we assume that w is split on X. The differential w being of the 1st kind gives that there is a Leray covering $\mathcal{U} = \{U_i\}$ of X where

$$w|_{U_i} = \phi_{1i} ... \phi_{mi}$$

with $\phi_{ki} \in H^0(U_i, \Omega^1_{X,cl})$. Since the differential w is split, we can order the $\{\phi_{ki}\}$ such that $\phi_{ki} \wedge \phi_{kj} = 0$ on the intersections $U_i \cap U_j$. Proposition 1.1 implies that on $U_i \cap U_j$

$$\phi_{ki} = c_{k,ij} \phi_{kj} \tag{1.5}$$

with $c_{k,ij} \in \mathbb{C}^*$ and $\prod_{k=1}^{m} c_{k,ij} = 1$. The m collections $\{c_{k,ij}\}$ for $k = 1, ..., m$ are elements in $Z^1(X, \mathbb{C}^*)$ and give rise to the rank 1 local systems which we denote by $\mathbb{C}_{\rho_k}$

and satisfy $\mathbb{C}_{\rho_1} \otimes ... \otimes \mathbb{C}_{\rho_m} \simeq \mathbb{C}$ (we remark that the isomorphism classes of these local systems are completely determined by w). It follows from (1.5) that each collection for a fixed k, $\{\phi_{ik}\}$, gives a section $\phi_k \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_k})$ and the result holds.♦

The presence of twisted closed holomorphic 1-forms $\phi_i \in H^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_{\rho_i})$ has implications on both the topology and geometry of the manifold X. On the topological side one observes that the cohomology exact sequence associated to the short exact sequence $0 \to \mathbb{C}_\rho \to \mathcal{O} \otimes \mathbb{C}_\rho \to \Omega^1_{X,cl} \otimes \mathbb{C}_\rho \to 0$ implies that $H^1(X, \mathbb{C}_\rho) \geq h^0(X, \Omega^1_{X,cl} \otimes \mathbb{C}_\rho)$ (hence in particular $\pi_1(X)$ must infinite). On the geometric side, if X is compact Kähler the presence of non-torsion, i.e. $L_\rho = \mathcal{O} \otimes \mathbb{C}_\rho$ non-torsion, twisted closed holomorphic 1-forms implies that X is fibered over curves of genus $g \geq 1$ as follows from the work of Beauville, Lazarsfeld-Green and Simpson (see [GrLa87],[Be92],[Si93] and [Ar92]).

2. Local structure of closed 2-differentials on surfaces

2.1 Differential operator for closed symmetric 2-differentials

A symmetric differential of degree 2 on a complex surface can be viewed as a generalized complex counterpart of a Riemannian metric on a real surface. We are going to use this fact to motivate the differential operator characterizing closed symmetric 2-differentials.

Let $w \in H^0(X, S^2\Omega^1_X)$ on a complex surface X. There is an open cover of $X, \mathcal{U} = \{U_i\}$ by local holomorphic charts, where

$$w|_{U_i} = a_i(z)(dz_{1i})^2 + b_i(z)dz_{1i}dz_{2i} + c_i(z)(dz_{2i})^2$$

On each of these open sets we get the holomorphic functions $detw|_{U_i} = a_i(z)c_i(z) - b_i(z)^2/4$ which together form an element of $H^0(X, \mathcal{O}(2K_X))$, called the discriminant of w.

Definition 2.1. *The discriminant divisor $Disc_w$ of $w \in H^0(X, S^2\Omega^1_X)$ is the divisor of zeros of the section $\{detw|_{U_i}\}_{i \in I}$ of $\mathcal{O}(2K_X)$. The core discriminant divisor of w is $Disc^0_w = Disc_w - 2(w)_0$.*

Geometrically, the support of Disc_w corresponds to the set of points where $w(x)$ either vanishes or defines one single line in T_xX. To better understand the support of Disc^0_w we give a new characterization of the divisor Disc^0_w. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of X such that $w|_{U_i} = h_i\hat{w}_i$ with $h_i \in \mathcal{O}(U_i)$ and $\hat{w}_i$ vanishes only in codimension 2. The divisor Disc^0_w can described via the local information $\mathrm{Disc}^0_w \cap U_i = \mathrm{Disc}_{\hat{w}_i}$. By definition $\mathrm{Supp}(\mathrm{Disc}^0_w) \subset \mathrm{Supp}(\mathrm{Disc}_w)$, moreover a point $x \in \mathrm{Supp}(\mathrm{Disc}_w) \setminus \mathrm{Supp}(\mathrm{Disc}^0_w)$ is such that $w(x) = 0$ but $\hat{w}_i(x)$ defines two distinct lines in T_xX. A more detailed characterization of the irreducible components of the divisors Disc_w and Disc^0_w is given in section 2.2.

Let $x \in X \setminus \mathrm{Disc}^0_w$, then the germ w_x of w at x splits, $w_x = \mu_1\mu_2$ with $\mu_i \in \Omega^1_{X,x}$. Say $x \in U_i$ with U_i as above, $w|_{U_i} = h_i\hat{w}_i$, then $x \in X \setminus \mathrm{Disc}^0_w$ implies that the discriminant of $\hat{w}_i$ does not vanish at x and therefore $\hat{w}_i$ (and hence w) splits at x, $\hat{w}_{ix} = \hat{\mu}_1\hat{\mu}_2$. Moreover, since the discriminant of $\hat{w}_i$ is nonzero at x, then $\hat{\mu}_1 \wedge \hat{\mu}_2(x) \neq 0$, which implies that x has a neighborhood U_x with holomorphic a chart (z_1, z_2) such that:

$$w|_{U_x} = g(z)dz_1dz_2 \tag{2.1}$$

with $g \in \mathcal{O}(U_x)$. The condition that $w|_{U_x}$ has a closed holomorphic decomposition, $w|_{U_x} = g(z)dz_1dz_2 = \mu_1\mu_2$ with μ_i closed holomorphic 1-forms, is equivalent to

$$g(z) = f_1(z_1)f_2(z_2) \tag{2.2}$$

($\mu_i \wedge dz_i \equiv 0$ implies that $\mu_i = f_i(z_i)dz_i$). The condition (2.2) can be characterized via the nonlinear differential equation $\frac{\partial^2 ln\ g(z)}{\partial z_1 \partial z_2} = 0$.

It follows from the Brioschi's formula for the Gaussian curvaure in terms of the 1st fundamental form [Sp99], that the differential operator giving the Gaussian curvature for a metric in the form $ds^2 = f(x)dx_1dx_2$ is $K(ds^2) = -\frac{2}{f}\frac{\partial^2 ln\ f(z)}{\partial x_1 \partial x_2}$. Hence the symmetric differential in (2.1) is closed if and only if $K_{\mathbb{C}}(w|_{U_x}) = 0$, where $K_{\mathbb{C}}$ is the operator obtained from K by replacing x_1, x_2 by z_1, z_2.

A symmetric 2-differential can not be put locally in the form (2.1) everywhere, but this is not a problem since the differential operator K for the Gaussian curvature works for metrics whose 1st fundamental form is arbitrary (works formally if the 1st fundamental form is degenerate, i.e. with discriminant zero at some points). Hence $K_{\mathbb{C}}$ works for any symmetric 2-differential (for a general symmetric 2-differential w, $K_{\mathbb{C}}(w)$ will be a meromorphic function with poles along the discriminant locus). If w is a symmetric 2-differential satisfying $K_{\mathbb{C}}(w) = 0$, then $K_{\mathbb{C}}(w|_{U_x}) = 0$ with $U_x \subset X \setminus \mathrm{Disc}^0_w$ as in 2.1 and hence it by our previous paragraph w is closed. Hence we obtain:

Proposition 2.1. *Let $w \in H^0(X, S^2\Omega^1_X)$ be of rank 2 on a X a connected complex surface, then w being closed is equivalent to:*

$$K_{\mathbb{C}}(w) = 0$$

Moreover, $K_{\mathbb{C}}(w) = 0$ is equivalent to w is of the 1st kind on $X \setminus Disc^0_w$.

2.2 Characterization of the breakdown locus B_w

In this section w is a closed symmetric 2-differential of rank 2. We start by showing that the breakdown locus B_w has no isolated points and then proceed to show that B_w is an analytic subvariety of codimension 1 and to characterize geometrically its components.

Lemma 2.2. *B_w has no isolated points.*

Proof. It is enough to show that if w is of the 1st kind in a punctured ball $\mathbb{B}^*$, then it is of the 1st kind on the whole ball $\mathbb{B}$. Let $w \in H^0(\mathbb{B}, S^2\Omega^1_{\mathbb{B}})$ be of the 1st kind on the punctured ball $\mathbb{B}^*$. According to corollary 1.2,

$$w|_{\mathbb{B}^*} = \phi_1\phi_2$$

with $(\phi_1, \phi_2) \in H^0(\mathbb{B}^*, \Omega^1_{X,cl} \otimes (\mathbb{C}_\rho \oplus \mathbb{C}^*_\rho))$. The triviality of the fundamental group of $\mathbb{B}^*$ implies that $\mathbb{C}_\rho \simeq \mathbb{C}^*_\rho \simeq \mathbb{C}$ and hence the ϕ_i can be chosen to be in $H^0(\mathbb{B}^*, \Omega^1_{X,cl})$. Again using $\pi_1(\mathbb{B}^*) = \{e\}$, it follows by integration that $\phi = df_i$ with $f_i \in \mathcal{O}(\mathbb{B}^*)$. Hartog's extension theorem implies that exist $\hat{f}_i \in \mathcal{O}(\mathbb{B})$ extending the f_i and hence w is of the 1st kind on $\mathbb{B}$ with $w = d\hat{f}_1 d\hat{f}_2$.♦

Proposition 2.1 tell us that that:

Corollary 2.2. *$B_w \subset Supp(Disc^0_w)$.*

To proceed we need to give a geometric description of the irreducible components of both discriminant loci. The support of the discriminant divisor decomposes into:

$$\mathrm{Supp}(\mathrm{Disc}_w) = N_w \cup S_w$$

where N_w and S_w are the union of all irreducible components of Disc_w of respectively odd and even multiplicities. The locus N_w corresponds to the points where w fails to split at. For the support of the core discriminant divisor we have:

$$\mathrm{Supp}(\mathrm{Disc}^0_w) = N_w \cup C_w \cup R_w$$

It follows from the characterization of Disc^0_w given after definition 2.1 that any $x \in \mathrm{Supp}(\mathrm{Disc}^0_w)$ is in the closure of the locus of points $y \in X$ such that $\hat{w}_i(y)$ defines a single line in T_yX (where $w|_{U_i} = h_i\hat{w}_i$ with $y \in U_i$, $h_i \in \mathcal{O}(U_i)$ and $\hat{w}_i$ is a symmetric 2-differential vanishing at most in codimension 2). Note that definition 2.1 gives directly that the divisor N_w is fully contained in $\mathrm{Supp}(\mathrm{Disc}^0_w)$, since only even multiples of irreducible components are subtracted from (Disc_w) to obtain (Disc^0_w).

The divisor C_w consists of the union of all irreducible components of $\mathrm{Supp}(\mathrm{Disc}_w)$ which are leaves simultaneously of the two foliations defined by the $\{\hat{w}_i\}_{i\in I}$ (a 2-differential defines two foliations where it splits). We will call the irreducible components of C_w the common leaves of w. The divisor R_w consists of all the irreducible components of $\mathrm{Supp}(\mathrm{Disc}^0_w)$ that are not in N_w or C_w. These will be the components for which at their general point x the two different foliations given by $\{\hat{w}_i\}_{i\in I}$ define leaves that are tangent at x but that do not coincide).

Theorem 2.4. *$B_w = N_w \cup C''_w$, where C''_w is a union of curves contained in C_w.*

Proof. The locus N_w is contained in B_w since the differential w splits on any $x \notin B_w$. Set $X' = X \setminus N_w$, $C'_w = C_w \cap X'$ and $R'_w = R_w \cap X'$ and get:

$$\mathrm{Supp}(\mathrm{Disc}^0_w) \cap X' = C'_w \cup R'_w$$

The desired result then follows if we show that the breakdown locus $B_{w|_{X'}}$ is an union of irreducible components of C'_w (with C''_w being the closure of this union).

By construction X' is the open subset of X where w is locally split. Hence given any $x \in X'$, there exists an open neighborhood U_x of x where $w|_{U_x} = \mu_1\mu_2$, $\mu_i \in H^0(U_x, \Omega^1_X)$. We can shrink U_x so that we can decompose $\mu_i = h_i\hat{\mu}_i$ with $h_i \in \mathcal{O}(U_x)$ and the $\hat{\mu}_i \in H^0(U_x, \Omega^1_X)$ are either nowhere vanishing or vanish only at x. Frobenius' theorem (if $\hat{\mu}_i(y) \neq 0$, then $\exists U_y$ open neighborhood of y where $\hat{\mu}_i = f_i du_i$, $f_i, u_i \in \mathcal{O}(U_y)$), then implies that the set $S \subset X'$ consisting of the points x where w fails to have a neighborhood U_x where $w|_{U_x} = g dz_1 dw_1$ with $g \in \mathcal{O}(U_x)$ and dz_1, dw_1 nowhere vanishing is discrete.

Consider the irreducible decomposition

$$C'_w \cup R'_w = \bigcup_{i=I} C'_{w,i} \cup \bigcup_{j=J} R'_{w,j}$$

where I, J are countable and $C'_{w,i}$ and $R'_{w,j}$ are the irreducible components of C'_w and R'_w respectively. Below, we will first show that the irreducible components $R'_{w,j}$ intersect $B_{w|_{X'}}$ only inside S, i.e. $R'_{w,j} \cap B_{w|_{X'}} \subset S \cup \bigcup_{i=I} C'_{w,i}$. Second, we will show that the irreducible components $C'_{w,i}$ are such that $C'_{w,i} \subset B_{w|_{X'}}$ or $C'_{w,i} \cap B_{w|_{X'}} \subset S$. These two results (and corollary 2.3) imply that $B_{w|_{X'}} = \bigcup_{i=I'} C'_{w,i} \cup S'$, with $S' \subset S$ and $I' \subset I$. The result then follows since $S' \subset \bigcup_{i=I'} C'_{w,i}$. The discreteness of S and $\bigcup_{i=I'} C'_{w,i}$ being an analytic subvariety of X' implies that if $x \in S'$ is not contained in $\bigcup_{i=I'} C'_{w,i}$, then x has a neighborhood U_x such that $U_x \cap B_w = x$, but by lemma 2.2 B_w has no isolated points.

Claim: $R'_{w,j} \cap B_{w|_{X'}} \subset S \cup \bigcup_{i=I} C'_{w,i}$

Before proceeding, note that by the definition of the set S it follows that any $x \in X' \setminus S$ has a neighborhood U_x with $g, z_1, z_2, w_1 \in \mathcal{O}(U_x)$ such that

$$w|_{U_x} = g dz_1 dw_1$$

and $\phi = (z_1, z_2) : U_x \to \Delta \times \Delta$, Δ a disc centered at 0, is a biholomorphism with $\phi(x) = (0, 0)$.

We will show that any $x \in R'_{w,j} \cap [X' \setminus (S \cup \bigcup_{i=I} C'_{w,i})]$ cannot lie in B_w.

Let U_x be a neighborhood of x as in the previous paragraph. Consider the leaf $L = \{z_1 = 0\}$ of w on U_x passing through x. By hypothesis x is not in a common leaf of w, hence L can not be a common leaf of w which implies that $L \not\subset \mathrm{Supp}(\mathrm{Disc}^0_w)$. If $L \subset \mathrm{Supp}(\mathrm{Disc}^0_w)$ then $dz_1 \wedge dw_1 = 0$ on L making L a leaf of dw_1 also, hence a common leaf for w. Hence $L \setminus [\mathrm{Supp}(\mathrm{Disc}^0_w) \cap L)] \neq \emptyset$

Pick $y \in L$ but not in $\mathrm{Supp}(\mathrm{Disc}^0_w)$, then by proposition 2.1 there is a (connected) neighborhood U_y of y where $w|_{U_y} = f(z_1)g(w_1)dz_1dw_1$ with $f, h \in \mathcal{O}(U_y)$ ($f(z_1)$ denotes a function $f(z_1, z_2)$ depending only on z_1). Let Δ' be a disc centered at 0 such that $\Delta' \times z_2(y) \subset \phi(U_y)$ and $W_x = z_1^{-1}(\Delta')(= \phi^{-1}(\Delta' \times \Delta))$. The function f has a clear holomorphic extension $\hat{f} \in \mathcal{O}(U_y \cup W_x)$, with $\hat{f}|_{W_x}(z_1, z_2) = f(z_1, z_2(y))$.

The same reasoning applied to h will not give an extension of h to $W_x \cup U_y$, so instead we use the extension of f and consider the function $\hat{h} = \frac{g}{\hat{f}}$. Clearly, $\hat{h}|_{U_y} = h$ hence $\hat{h}$ is a function of w_1 alone. The function $\hat{h}|_{W_x}$ is holomorphic since the irreducible components of the polar divisor $(\hat{h}|_{W_x})_\infty$ if they exist must be some of the irreducible components of the divisor of zeros of $\hat{f}$ which will be a union of curves $\{z_1 = c\}$ and hence intersect non-trivially U_y but this intersection must be empty since $h|_{U_y} = h$ is holomorphic.

It follows from the previous two paragraphs that the closed holomorphic decomposition of w at U_y, $w|_{U_y} = f(z_1)h(w_1)dz_1dw_1$, propagates to give the closed holomorphic decomposition on the neighborhood W_x of x, $w|_{W_x} = \hat{f}(z_1)\hat{h}(w_1)dz_1dw_1$, making $x \notin B_w$.

Claim: $C'_{w,i} \subset B_{w|_{X'}}$ or $C'_{w,i} \cap B_{w|_{X'}} \subset S$.

In addition to the properties, described two paragraphs above, that we can guarantee for an open neighborhood U_x of $x \in X' \setminus S$, we can equally guarantee the existence of an open neighborhood $U'_x \subset U_x$ and $w_2 \in \mathcal{O}(U'_x)$ such that $\phi' = (w_1, w_2) : U'_x \to \Delta' \times \Delta'$ is a biholomorphism.

Consider the subsets $C^*_{w,i} = C'_{w,i} \cap (X' \setminus S)$ and $V_i = C^*_{w,i} \cap (X \setminus B_w)$. The set $C^*_{w,i}$ is connected since by the local parametrization theorem [De12] an irreducible component of an analytic variety punctured by a discrete set is connected. The subset V_i consisting of all points of $C^*_{w,i}$ where w has a local holomorphic decomposition is clearly open in $C^*_{w,i}$. We proceed to show that V_i is also closed in $C^*_{w,i}$. Since $C^*_{w,i}$ is connected, V_i being both open and closed implies the desired result that the irreducible components $C'_{w,i}$ are such that $C'_{w,i} \subset B_{w|_{X'}}$ (when $V_i = \emptyset$ and use B_w closed) or $C'_{w,i} \cap B_{w|_{X'}} \subset S$ (when $V_i = C^*_{w,i}$).

Let $x \in C^*_{w,i}$ be an accumulation point of V_i. Pick $y \in V_i \cap U'_x$, with U'_x as in two paragraphs above. Since $C^*_{w,i}$ is a common leaf of w, $y \in L_x = \{z_1 = 0\} = \{w_1 = 0\}$. Hence y has a neighborhood U_y such that $\phi(U_y) \supset \Delta'' \times z_2(y)$ and $\phi'(U_y) \supset \Delta'' \times w_2(y)$, Δ'' a disc centered at 0, where $w|_{U_y} = g dz_1 dw_1 = f(z_1)h(w_1)dz_1dw_1$ with $f, h \in \mathcal{O}(U_y)$. The functions f and h are clearly extendable to $\hat{f} \in \mathcal{O}(z_1^{-1}(\Delta'') \cup U_y)$

and $\hat{h} \in \mathcal{O}(w_1^{-1}(\Delta'') \cup U_y)$. By construction $W_x = z_1^{-1}(\Delta'') \cap w_1^{-1}(\Delta'')$ is a connected open set containing x and y and $g|_{W_x \cap U_y} = \hat{f}\hat{h}|_{W_x \cap U_y}$, hence $g|_{W_x} = \hat{f}\hat{h}|_{W_x}$ giving a holomorphic decomposition of w on the neighborhood W_x of x, i.e. $x \in V_i$.

2.3 Monodromy at B_w

Let $w \in H^0(X, S^2\Omega^1_X)$ be closed of rank 2 and $B_w = \sum_{j \in J}$, J countable, be the irreducible decomposition of the breakdown locus B_w. Let $\mathcal{U} = \{U_i\}$ be a Leray covering of $X \setminus B_w$ where

$$w|_{U_i} = \phi_{1i}\phi_{2i}$$

with $\phi_{ki} \in H^0(U_i, \Omega^1_{X,cl})$, $k = 1, 2$. The abelian rank of a closed symmetric 2-differential of rank 2 is trivial, it follows then from proposition 1.1 that if $U_i \cap U_j \neq \emptyset$, then

$$\begin{bmatrix} \phi_{1i} \\ \phi_{2i} \end{bmatrix} = g_{ij} \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \end{bmatrix}$$

with $g_{ij} \in G = \{\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}, \begin{bmatrix} 0 & c \\ c^{-1} & 0 \end{bmatrix} \mid c \in \mathbb{C}^*\}$. The collection $\{g_{ij}\}$ gives a 1-cocycle with values in the group G, i.e. $\{g_{ij}\} \in Z^1(\mathcal{U}, G)$. Hence given $x_0 \in X \setminus B_w$, we obtain a representation $\rho : \pi_1(X \setminus B_w, x_0) \to G$.

If w is split, then $\mathrm{Im}\rho \subset G'$, with $G' = \{\begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}, \forall c \in \mathbb{C}^*\} \subset G$ (this follows from being able to get a consistent ordering of the foliations on all the U_i). Since G' is abelian we get a representation, $\rho_w : \pi_1(X \setminus B_w) \to G'$, that is independent of the base point and factors through $H_1(X \setminus B_w, \mathbb{Z})$, and gives:

$$\bar{\rho}_w : H_1(X \setminus B_w, \mathbb{Z}) \to G'$$

Associated with each irreducible component B_j, let $\gamma_i j \in H_1(X \setminus B_w, \mathbb{Z})$ be the class of a simple loop around $B_i j$ (boundary to a disc transversal to B_j centered at a general point of B_j) which can have either orientation.

Definition 2.2. *Let $w \in H^0(X, S^2\Omega^1_X)$ be split, closed of rank 2 and $B_w = \sum_{j \in J} B_j$, J countable, be the irreducible decomposition of the breakdown locus B_w. To each irreducible component B_j we associate the monodromy index $M(B_j, w) = \{c, c^{-1}\}$, if $\bar{\rho}_w(\gamma_j) = \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix}$, with $\bar{\rho}_w$ and γ_j as above.*

2.4 Local form at B_w

The goal of this section and the main result of this article is to give the general form of a split closed symmetric 2-differential of rank 2 w at the general point of an irreducible component of the breakdown locus B_w. We will see that the closed decompositions of w can acquire essential singularities and have non-trivial monodromy at the breakdown locus B_w. We will show that the essential singularities have an algebraic feature, they come from exponential functions with meromorphic functions with poles along B_w as exponents. Moreover, we will give a bound on the order of the poles of the meromorphic functions appearing as exponents. The bounds come from the order of contact of the two foliations of w along the irreducible components of B_w.

We start with some examples of closed symmetric 2-differentials for which B_w is non-empty.

Example: (non-split) Let z_1 be a holomorphic coordinate of $\mathbb{C}^n$ and $f \in \mathcal{O}(\mathbb{C}^n)$, set $w = z_1(dz_1)^2 - (df)^2$. The differential is non split at all points in $\{z_1 = 0\}$ but it is closed since any point $y \in X \setminus \{z_1 = 0\}$ has a neighborhood U_y where $\sqrt{z_1}$ exists and hence w has a holomorphic exact decomposition $w|_{U_y} = d(\frac{2}{3}z_1^{\frac{3}{2}} + f)d(\frac{2}{3}z_1^{\frac{3}{2}} - f)$.

If the differential is locally split at x, then a 2nd layer of the failure of w to have a holomorphic closed decomposition at x is due to the monodromy in the factors of the closed decompositions (not the monodromy of the foliations) around B_w.

Example (monodromy of the closed decompositions): Let $B \subset \mathbb{C}^2$ be a sufficiently small open ball about the origin where $1 + z_2$ is invertible. Consider $w = (1 + z_2)^\alpha dz_1 d[z_1(1 + z_2)]$. Recall that the differential w has a holomorphic closed decomposition at a point $x \in B$ if and only if we can decompose $(1 + z_2)^\alpha$ as a product of holomorphic functions of z_1 and $z_1(1 + z_2)$ near x. At points in the complement of $\{z_1 = 0\}$ we have the decomposition $(1 + z_2)^\alpha = z_1^{-\alpha}[z_1(1 + z_2)]^\alpha$, but at points in $\{z_1 = 0\}$ the functions involved are multivalued, hence no holomorphic closed decomposition of w is possible at $x \in \{z_1 = 0\}$. In fact this monodromy is infinite if $\alpha \notin \mathbb{Q}$, meaning that even after finite ramified coverings the symmetric differential would not have an exact decomposition along the pre-image of $\{z_1 = 0\}$.

If the differential is both locally split at x and no monodromy occurs, then w a 3rd level of failure to have a holomorphic closed decomposition is due to the singularities of the 1-differentials on the decomposition.

Example: (meromorphic singularities) $w = (dz_1)^2 + z_1 z_2 dz_1 dz_2 = dz_1(dz_1 + z_1 z_2 dz_2)$ has the common leaf $L = \{z_1 = 0\}$. The differential w is closed because the 1-form $dz_1 + z_1 z_2 dz_2$ has an integrating factor, $\frac{1}{z_1}$, which is a function of z_1. This integrating factor produces the closed meromorphic decomposition $w = d(\frac{z_1^2}{2})(\frac{1}{z_1}dz_1 + z_2dz_2)$. Note that since the abelian rank of w is trivial any other closed decomposition of w would differ just by multiplicative constants hence meromorphic singularities would be always present in the closed decompositions of w.

Example: (essential singularities) This example shows that even essential singularities can occur, $w = e^{\frac{z_2}{1+z_1z_2}}dz_1d[z_1(1+z_1z_2)]$. The 1-differentials in the split closed decomposition are unique up to multiplicative constants, as it was shown in proposition 1.1, and the constants will cancel each other so in fact the decomposition is unique and has the form

$$w = e^{\frac{z_2}{1+z_1z_2}}dz_1d[z_1(1+z_1z_2)] = e^{\frac{1}{z_1}}dz_1e^{-\frac{1}{z_1(1+z_1z_2)}}d[z_1(1+z_1z_2)]$$

with essential singularities occurring on the closed 1-forms at $\{z_1 = 0\}$.

Lemma 2.5. *Let X be a complex 2-manifold, $w \in H^0(X, S^2\Omega^1_X)$ be split of rank 2 and L be an irreducible component of a common leaf of w. Then there is an $m \in \mathbb{N}$ such that the general point x of L has a neighborhood U_x with a holomorphic chart (z_1, z_2) where*

$$w|_{U_x} = f(z_1, z_2)dz_1d[z_1(1 + z_1^m z_2)]$$

Proof. The differential w being split implies that every point $x \in X$ has an open neighborhood U_x such that $w|_{U_x} = \mu_1\mu_2$, $\mu_i \in H^0(X, \Omega^1_X)$. By shrinking U_x we can factor the 1-forms μ_i in the form $\mu_i = f_i\hat{\mu}_i$ with $f_i \in \mathcal{O}(U_x)$ and $\hat{\mu}_i$ either non-vanishing or vanishing only at x. Since by Frobenius theorem a non-vanishing 1-form in dimension 2 is integrable, it follows that there is a discrete set $S \subset X$ such that all $x \in X \setminus S$ have a neighborhood U_x where

$$w|_{U_x} = hdvdr \tag{2.3}$$

with $h, v, r \in \mathcal{O}(U_x)$, $v(x) = r(x) = 0$, dv and dr nowhere zero on U_x.

Let x be a general point of L, using the notation of (2.3) we have $L \cap U_x = \{v = 0\} = \{w = 0\}$ with:

$$r = vu$$

with u a unit on U_x. After shrinking U_x we can assume there is a holomorphic local chart on U_x, (v_1, v_2) such that $v_1 = v$. Consider the series expansion $u(v_1, v_2) = \sum_{i=0,j=0}^{\infty} c_{ij}v_1^iv_2^j$ and let $m = \min\{i | \exists j > 0 \text{ s.t. } c_{ij} \neq 0\}$ (the Taylor series of u must involve v_2 since $dv \wedge dr \not\equiv 0$). Then decompose u as

$$u(v_1, v_2) = s(v_1) + v_1^m(t(v_2) + v_1g(v_1, v_2))$$

where $s(v_1) = \sum_{i=0}^{\infty} c_{i0} v_1^i$ is a holomorphic function in v_1 with $s(0) \neq 0$ and hence a unit in a neighborhood of 0. Note $t(0) = 0$ and more importantly $t(v_2)$ is not constant. Hence $dt(v_2)$ is non vanishing at the general point of $L \cap U_x$. If $dt(v_2)(x) = 0$, then change x to make $dt(v_2)(x) \neq 0$.

Set $z_1 = v_1 s(v_1)$ and $z_2 = \frac{t(v_2)+v_1 g(v_1,v_2)}{s(v_1)^{m+1}}$. By construction $dz_1(x) \neq 0$, $dz_2(x) \neq 0$ and $dz_1 \wedge dz_2(x) \neq 0$ and

$$r = z_1(1 + z_1^m z_2)$$

giving the desired $w|_{U_x} = f(z_1, z_2) dz_1 d[z_1(1 + z_1^m z_2)]$ with $f = \frac{h}{s(v_1)+v_1 s'(v_1)}$.

Observing that $m = \mathrm{ord}_{\{v_1=0\}}(\frac{\partial r}{\partial v_2}) - 1$, it follows that m is independent of the choice of v and r with dv and dr non-vanishing such that $w = hdvdr$ and the choice of holomorphic chart (v_1, v_2) with $v_1 = v$. The independence of m on the above choices plus the connectedness of L minus a discrete set of points implies that any other general point of L would give the same m and hence m is naturally associated to the irreducible component L.♦

Definition 2.3. *An irreducible component L of a common leaf of $w \in H^0(X, S^2\Omega_X^1)$ of rank 2 is said to have order of contact m, $O(L, w) = m$, if in a neighborhood U_x of the general point $x \in L$ w is of the form as in lemma 2.5, i.e. $w|_{U_x} = f(z_1, z_2) dz_1 d[z_1(1 + z_1^m z_2)]$.*

Theorem 2.6. *Let X be a complex 2-manifold, $w \in H^0(X, S^2\Omega_X^1)$ be split, closed of rank 2 and L an irreducible component of a common leaf of w. Then the general point x in L has a neighborhood U_x where $w|_{U_x}$ has a decomposition of the form:*

$$w|_{U_x} = z_1^k (1 + z_1^m z_2)^{\alpha} e^{f(z_1)} e^{g(z_1(1+z_1^m z_2))} dz_1 d[z_1(1 + z_1^m z_2)]$$

where:

i) $m = O(L, w)$, $k = ord_L(w)_0$ ($(w)_0$ is the divisorial zero of w) and $\alpha = \frac{\log c}{2\pi i} + k$ for some $k \in \mathbb{Z}$ and $c \in M(L, w)$.

ii) f and g are meromorphic functions on Δ^* with poles of order at most m at 0.

Remark: The local form of $w|_{U_x}$ in the theorem can be rewritten as the following decomposition of $w|_{U_x}$ as the product of two closed 1-differentials (in a generalized sense since they might be multi-valued) with singularities along L:

$$w|_{U_x} = (z_1^{\beta} e^{f(z_1)} dz_1)([z_1(1 + z_1^m z_2)]^{\alpha} e^{g(z_1(1+z_1^m z_2))} d[z_1(1 + z_1^m z_2)])$$

with $\alpha + \beta = \mathrm{ord}_L(w)_0$ and α, f and g as in the theorem.

Proof. According to the lemma 2.5 the general point $x \in L$ has a neighborhood U_x with a holomorphic coordinate chart (z_1, z_2) such that $x = (0,0)$, $L \cap U_x = \{z_1 = 0\}$ and $w|_{U_x} = v(z_1, z_2)dz_1 d[z_1(1 + z_1^m z_2)]$ with $v \in \mathcal{O}(U_x)$.

We claim that if we shrink U_x the divisor of zeros of $w|_{U_x}$ is $(v)_0 = kL$ and hence

$$w|_{U_x} = z_1^k \tilde{w}$$

with $\tilde{w} \in H^0(U_x, S^2\Omega_X^1)$ a nowhere vanishing closed symmetric differential of the form:

$$\tilde{w} = \tilde{v}(z_1, z_2)dz_1 d[z_1(1 + z_1^m z_2)] \tag{2.4}$$

with $\tilde{v}(z_1, z_2) \in \mathcal{O}^*(U_x)$.

By shrinking U_x we can make $(v)_0$ a finite union of irreducible components all passing through x. The differential w being closed implies (theorem 2.4) that all $y \in U_x \setminus L$ have a neighborhood U_y such that $v|_{U_y} = f(z_1)g(z_1(1 + z_1^m z_2))$. This implies that if an irreducible component of $(v)_0$ is not L, then it must be a level set of z_1 or $z_1(1 + z_1^m z_2)$ not passing through x, a contradiction. It follows then that $(v)_0 = kL$ for some $k \in \mathbb{N}$ and (2.4) holds.

Note that we have the equality $M(L, \tilde{w}) = M(L, w)$, this can be seen for example by noting that the factor on the local holomorphic decompositions of w and $\tilde{w}$ corresponding to the foliation $d[z_1(1 + z_1^m z_2)]$ does not change (the 1-cocycle with values in $\mathbb{C}^*$ corresponding to this foliation remains unchanged) hence the index remains unchanged.

The neighborhood U_x can be chosen to be the bi-disc $U_x = \Delta_{\epsilon_1} \times \Delta_{\epsilon_2}$, $\epsilon_i > 0$, relative to the coordinate chart (z_1, z_2). On U_x we have two maps $\pi_1 : U_x \to \mathbb{C}$ given by $\pi_1(z_1, z_2) = z_1$ and $\pi_2 : U_x \to \mathbb{C}$ given by $\pi_2(z_1, z_2) = z_1(1 + z_1^m z_2)$.

Let $\mathcal{U} = \{U_i\}_{i=1,\ldots,k}$, $k \in \mathbb{N}$, be a Leray open covering of the punctured disc $\Delta_{\epsilon_1}^*$. The Leray covering $\{U_i \times \Delta_{\epsilon_2}\}_{i=1,\ldots,k}$ of $U_x \setminus \{z_1 = 0\}$ is such that one has the holomorphic closed decompositions on its open sets:

$$\tilde{w}|_{U_i \times \Delta_{\epsilon_2}} = \breve{f}_i(z_1)\breve{g}_i(z_1(1 + z_1^m z_2))dz_1 d[z_1(1 + z_1^m z_2)] \tag{2.5}$$

where $\breve{f}_i = \pi_1^* f_i$ with $f_i \in \mathcal{O}(U_i)$ and $\breve{g}_i = \pi_2^* g_i$ with $g_i \in \mathcal{O}(U_i')$ ($U_i \subset U_i' = \pi_2(U_i \times \Delta_{\epsilon_2})$). The existence of such closed decomposition on the whole open sets $U_i \times \Delta_{\epsilon_2}$ is guaranteed since the open sets are simply connected and the fibers of both $\pi_1|_{U_i \times \Delta_{\epsilon_2}}$ and $\pi_2|_{U_i \times \Delta_{\epsilon_2}}$ are connected (assuming ϵ_1 and ϵ_2 are sufficiently small).

Since w is a symmetric differential of degree 2 and rank 2, the abelian rank of w is trivial which due to proposition 1.1 implies that on the intersections $U_i \cap U_j$:

$$f_i = c_{ij} f_j \qquad g_i = c_{ij}^{-1} g_j \tag{2.6}$$

The collection $\{c_{ij}\}$ defines a 1-cocycle in $Z^1(\mathcal{U}, \mathbb{C}^*)$ defining a representation of $\rho : \pi_1(\Delta^*_{\epsilon_1}) \to \mathbb{C}^*$. There is a natural homomorphism $\phi_L : \pi_1(\Delta^*_{\epsilon_1}) \to H_1(X \backslash B_w, \mathbb{Z})$ sending the class of a simple loop γ around the origin oriented positively to the class of a simple loop γ_L around the irreducible component L (as in definition 2.2). By construction, $\rho(\gamma)$ is one of the diagonal entries of $\bar{\rho}_w(\gamma_L)$, i.e. $\rho(\gamma) \in \mathrm{M(L,w)} = \{c, c^{-1}\}$.

To simplify notation rescale the coordinates so that $U_x = \Delta \times \Delta$, Δ the unit disc centered at 0 and set the covering $\mathcal{U}$ of Δ^* to be $\{U_{-1}, U_0, U_1\}$ with $U_i = (0,1) \times (\frac{(2i-1)}{3}\pi - \epsilon, \frac{(2i+1)}{3} + \epsilon)$, $\epsilon > 0$ sufficiently small, if expressed in polar coordinates.

Consider the universal covering map $e : \mathcal{H}^- \to \Delta^*$, $z \to e^z$, with $\mathcal{H}^- = \{z \in \mathbb{C} | \mathrm{Re} z < 0\}$, and the open covering of $\mathcal{H}^-$, $\tilde{\mathcal{U}} = \{\tilde{U}_j\}_{j \in \mathbb{Z}}$ where the $\tilde{U}_j = (-\infty, 0) \times (\frac{(2j-1)}{3}\pi - \epsilon, \frac{(2j+1)}{3} + \epsilon)$. Note that $e : \tilde{U}_j \to U_{[j]}$, with $[j] \in \{-1, 0, 1\}$ and $j \equiv [j] \mod 3$, is a biholomorphism.

Let $\{\tilde{f}_j\}_{j \in \mathbb{Z}} \in C^0(\tilde{\mathcal{U}}, \mathcal{O}^*)$ be the 0-cochain defined by $\tilde{f}_j = f_{[j]} \circ e \in \mathcal{O}^*(\tilde{U}_j)$. The coboundary $\delta\{\tilde{f}_j\}$ gives a 1-cocycle with values in $\mathbb{C}^*$, $\{\tilde{c}_{jj'}\} \in Z^1(\tilde{\mathcal{U}}, \mathbb{C}^*)$, since $\tilde{f}_j = \tilde{c}_{jj'} \tilde{f}_{j'}$ on $\tilde{U}_j \cap \tilde{U}_{j'}$. The space $\mathcal{H}^-$ being simply connected implies that $\{\tilde{c}_{jj'}\} \in B^1(\tilde{\mathcal{U}}, \mathbb{C}^*)$. Hence there is a collection $\{\tilde{c}_j\} \in C^0(\tilde{\mathcal{U}}, \mathbb{C}^*)$ such that $\tilde{c}_{j'} \tilde{f}_j = \tilde{c}_j \tilde{f}_j$ on $\tilde{U}_j \cap \tilde{U}_{j'}$ giving:

$$\{\tilde{c}_j \tilde{f}_j\} =: F \in \mathcal{O}^*(\mathcal{H}^-) \tag{2.7}$$

The function F, due to $\tilde{c}_{jj'} = c_{[j][j']}$ and the discussion following (2.6), satisfies the special transformation law

$$F(z + 2\pi i) = \rho(\gamma) F(z)$$

Since $e^{(\frac{\log \rho(\gamma)}{2\pi i})z}$ is function with the same transformation law as F, it follows that:

$$F = e^{(\frac{\log \rho(\gamma)}{2\pi i})z} \hat{f}(e^z)$$

with $\hat{f} \in \mathcal{O}^*(\Delta^*)$.

The above implies that if we set $c_i = \tilde{c}_i$, $i = -1, 0, 1$, then

$$c_i f_i|_{\hat{U}_i} = (z^{\frac{\log \rho(\gamma)}{2\pi i}} \hat{f})|_{\hat{U}_i}$$

with $z^{\frac{\log \rho(\gamma)}{2\pi i}}$ representing the principal branch of the power function and $\hat{U}_i = (0,1) \times (\frac{(2i-1)}{3}\pi, \frac{(2i+1)}{3}\pi)$ (if expressed in polar coordinates). The same reasoning can be done with respect to the collection $\{g_i\}_{i=-1,0,1}$ using the collection $\{c_i^{-1}\}_{i=-1,0,1} \in C^0(\mathcal{U}, \mathbb{C}^*)$ to obtain

$$c_i^{-1} \; g_i|_{\hat{U}_i} = (z^{-\frac{\log \rho(\gamma)}{2\pi i}} \hat{g})|_{\hat{U}_i}$$

with $\hat{g} \in \mathcal{O}^*(\Delta^*)$ and $z^{-\frac{\log \rho(\gamma)}{2\pi i}}$ representing the principal branch of power function.

Finally, using the above descriptions of the collections $\{c_i\ f_i\}$ and $\{c_i^{-1} g_i\}$ it follows that we can rewrite the local holomorphic closed decompositions described in (2.5) by changing $\breve{f}_i$ and $\breve{g}_i$ to respectively $c_i \breve{f}_i$ and $c_i^{-1} \breve{g}_i$ and obtain the global decomposition of $\tilde{w}$ on $\Delta^* \times \Delta$:

$$\tilde{w}|_{\Delta^* \times \Delta} = (1 + z_1^m z_2)^{-\frac{\log \rho(\gamma)}{2\pi i}} \hat{f}(z_1) \hat{g}(z_1(1 + z_1 z_2)) dz_1 d[z_1(1 + z_1^m z_2)] \tag{2.8}$$

recall that by construction $\rho(\gamma) \in M(L, w)$. Note that behind the global decomposition (2.8) there is a closed decomposition of $\tilde{w}$ on U_x but it involves the multi-valued functions and functions with singularities along L, $\tilde{w} = (z_1^{\frac{\log \rho(\gamma)}{2\pi i}} \hat{f}(z_1) dz_1)([z_1(1 + z_1^m z_2)]^{-\frac{\log \rho(\gamma)}{2\pi i}} \hat{g}(z_1(1 + z_1 z_2)) d[z_1(1 + z_1^m z_2)])$.

The next goal is to understand the singularities that are possible for the functions $\hat{f}, \hat{g} \in \mathcal{O}^*(\Delta^*)$. To achieve this goal, we use the fact that the product of $\hat{f}(z_1)$ with $\hat{g}(z_1(1 + z_1^m z_2))$ extends to a holomorphic function on $\Delta \times \Delta$ since it satisfies:

$$\hat{f}(z_1) \hat{g}(z_1(1 + z_1^m z_2))|_{\Delta^* \times \Delta} = \hat{v}(z_1, z_2)|_{\Delta^* \times \Delta} \tag{2.9}$$

where $\hat{v}(z_1, z_2) = \tilde{v}(z_1, z_2)(1 + z_1^m z_2)^{\frac{\log \rho(\gamma)}{2\pi i}} \in \mathcal{O}^*(\Delta \times \Delta)$.

The functions $\hat{f}$ and $\hat{g}$ do not necessarily have a well defined logarithm on Δ^*, since $f_*, g_* : \pi_1(\Delta^*) \to \pi_1(\mathbb{C}^*)$ are not necessarily trivial. However, if we set $k_1 = f_*(\gamma) \in \pi_1(\mathbb{C}^*) = \mathbb{Z}$ and $k_2 = g_*(\gamma) \in \pi_1(\mathbb{C}^*) = \mathbb{Z}$ with γ a simple loop around 0 positively oriented, then $\hat{f}(z) = z^{k_1} \breve{f}(z)$ and $\hat{g}(z) = z^{k_2} \breve{g}(z)$ are such that the functions $\breve{f}, \breve{g} \in \mathcal{O}^*(\Delta^*)$ have well defined logarithmic functions, $f = \log \breve{f}, g = \log \breve{g} \in \mathcal{O}(\Delta^*)$.

It follows from (2.9) that $\hat{f}(z)\hat{g}(z) = \hat{v}(z, 0)$ and hence $\hat{f}(z)\hat{g}(z) \in \mathcal{O}(\Delta^*)$ extends to a holomorphic function on Δ which forces $k_2 = -k_1$ ($(\hat{f}\hat{g})_* : \pi_1(\Delta^*) \to \pi_1(\mathbb{C}^*)$ is trivial since it factors through $\hat{v}(z, 0)_* : \pi_1(\Delta) \to \pi_1(\mathbb{C}^*)$ and $(\hat{f}\hat{g})_*(\gamma)) = f_*(\gamma) + g_*(\gamma)$). This implies that decomposition (2.8) can be rewritten as:

$$\tilde{w} = (1 + z_1^m z_2)^{-\frac{\log \rho(\gamma)}{2\pi i} - k_1} e^{f(z_1)} e^{g(z_1(1 + z_1 z_2))} dz_1 d[z_1(1 + z_1^m z_2)] \tag{2.10}$$

We are now interested in the singularities of $f, g \in \mathcal{O}(\Delta^*)$. It follows from (2.9) that:

$$f(z_1) + g(z_1(1 + z_1^m z_2)) = \log \hat{v}(z_1, z_2) \tag{2.11}$$

where $\log \hat{v}(z_1, z_2) \in \mathcal{O}(U_x)$. To derive conditions on $f, g \in \mathcal{O}(\Delta^*)$ from (2.11) consider the Laurent series expansions:

$$f(z_1) = \sum_{i=-\infty}^{\infty} a_i z_1^i$$

$$g(z_1(1+z_1^m z_2)) = \sum_{i=-\infty}^{\infty} b_i[z_1(1+z_1^m z_2)]^i$$

for the sum $f(z_1)+g(z_1(1+z_1^m z_2))$ to be holomorphic we must have

$$\sum_{i=-\infty}^{-1} a_i z_1^i + \sum_{i=-\infty}^{-1} b_i[z_1(1+z_1^m z_2)]^i =: r(z_1,z_2) \tag{2.12}$$

with $r(z_1,z_2)$ holomorphic. To simplify our notation, we quickly note that for $r(z_1,0)$ to be holomorphic we must have $b_i=-a_i$ $\forall i<0$ from which it follows that

$$r(z_1,z_2) = \sum_{i=-\infty}^{-1} a_i z_1^i[1-(1+z_1^m z_2)^i]$$

Consider the expansion $1-(1+z_1^m z_2)^i = \sum_{k=1}^{\infty} c_k^{(i)} z_1^{km} z_2^k$. Of the coefficients $c_k^{(i)}$ we will only use the fact that they are all non-vanishing and $r(z_1,z_2) = \sum_{i=-\infty}^{-1}\sum_{k=1}^{\infty} a_i c_k^{(i)} z_1^{i+km} z_2^k$.

Reorganizing the terms of the last expansion of $r(z_1,z_2)$, one obtains:

$$r(z_1,z_2) = \sum_{j=-\infty}^{\infty}\Big(\sum_{k=\min\{1,\ulcorner \frac{j}{m}\urcorner\}}^{\infty} a_{j-km} c_k^{(j-km)} z_2^k\Big) z_1^j$$

The holomorphicity of $r(z_1,z_2)$ implies that $\forall j\le -1$ the functions

$$s_j(z_2) = \sum_{k=1}^{\infty} a_{j-km} c_k^{(j-km)} z_2^k$$

must vanish, which using the non vanishing of the $c_k^{(i)}$ implies that

$$a_i = 0 \qquad\qquad \forall i < -m$$

this jointly with the equality $b_i=-a_i$ $\forall i<0$ give the desired result ii) stating that f and g are meromorphic functions with poles of order at most m at the origin.♦

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