SAMPLE EXAM MATH 210 (Spring 2008)

Name:

The first 9 problems are worth 10 points each. The last two problems are 5 points each.

Problem 1: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 5 \\ 2 & 5 & 1 \end{bmatrix}$ and use it to solve $A\mathbf{x} = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$.

Solution:

$$A^{-1} = \begin{bmatrix} -19 & 8 & -2 \\ 7 & -3 & 1 \\ 7 & -1 & 0 \end{bmatrix}$$

The solution to the system is:

$$\mathbf{x} = \begin{bmatrix} -19 & 8 & -2 \\ 7 & -3 & 1 \\ 7 & -1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix}$$

Problem 2: Find the complete solution to:

$$\begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 6 & 4 & 1 \\ 3 & 9 & 6 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$$

Solution:

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} + y \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Problem 3: a) Let $V = Span(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix})$.

a) Find a basis for V.

b) What is the dimension of V?

c) Find 3 vectors, among the six vectors above, that are linearly dependent. Justify your answer.

Solution:

a) $\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}\}$, note that there are many other possible answers.

b) The dimension of V is 3 (3 vectors in the basis)

c) For example $\{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}\}$

Since $-\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Problem 4: Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 3 & 3 & 3 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

a) Find the rank of A.

b) Find a basis of the row space of A, R(A).

c) Find a basis of $R(A)^{\perp}$, the orthogonal complement of R(A).

Solution:

a) rk A=3

b) $\{[1212], [2121], [1101]\}$, again many possible answers here.

c) $\{\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\}$, any other answer must be a nonzero multiple of this vector.

Problem 5: Write down one matrix A for each of the following conditions:

a) $A = A^T$ and det(A)=6

b) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for C(A) and $\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for N(A).

c) C(A) is a subspace of $\mathbf{R}^4$ of dimension 2 and $\dim \mathrm{N(A)} = 1$.

d) $A \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$.

e) A is is not diagonalizable and invertible.

Solution:

a) A=[6], many other possible solutions.

b) $A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, many other possibilities but I have seen many wrong examples from students.

c) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, many other possibilities.

d) $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & -1 & 2 \end{bmatrix}$, many other possibilities.

e) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, many other possibilities.

Problem 6: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ -1 & 1 \end{bmatrix}$.

a) Find the point $\mathbf{p}$ in the column space $C(A)$ closest to $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

b) What is the best approximate solution to the linear system $A\mathbf{x} = \mathbf{b}$ with A and $\mathbf{b}$ as above?

c) Find the projection matrix P for the column space $C(A)$.

Solution:

I will first solve b)

b) To find $\mathbf{x}$ solve the associated normal system $A^T A\mathbf{x} = A^T\mathbf{b}$.

a) $\mathbf{p} = A\mathbf{x}$, where $\mathbf{x}$ is the 2-vector found in b) above.

c) $P = A(A^T A)^{-1} A^T$

Problem 7: a) Using the cofactor formula find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ c & 0 & 2 & 1 \\ 0 & c & 0 & 1 \end{bmatrix}$$

b) For which c is A invertible? Explain.

Solution:

a) Det A=c(c-1).

b) A is invertible if determinant of A is $\neq 0$, hence if $c \neq 0, 1$.

Problem 8: Consider the matrix

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$$

a) What is the matrix A^{10}?

b) What is the $\lim_{k\to\infty} A^k$?

Solution:

a) Eigenvalues are $\lambda = 1$ and $\lambda = .4$

$$A^{10} = -\frac{1}{3}\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (.4)^{10} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$$

b) $\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

Problem 9: a) Find the general solution to the following system of differential equations:

$$\frac{dx}{dt} = 4x + 3y$$
$$\frac{dy}{dt} = -2x - 3y$$

b) Describe all initial conditions $(x(0), y(0))$ for which solutions satisfying those initial conditions are such that $\lim_{t\to+\infty}(x(t), y(t)) = (0, 0)$.

Solution:

a) Eigenvalues are $\lambda = -2$ and $\lambda = 3$ and the general solution is:
$\mathbf{x}(t) = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

b) The initial conditions that satisfy the requirements are the ones that force $c_2 = 0$, this is achieved by setting the initial conditions $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, where a can be any number.

Problem 10: Let A be a 3×3 matrix with a factorization $A = S\Lambda S^{-1}$, where $S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$. **Answer the following questions without finding out what the matrix A is.**

a) Which of the vectors $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$ is an eigenvector of A? Explain.

b) What is the vector $A^2 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$?

c) Given a vector $\mathbf{v}$, we form a sequence $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v_2}, ..., A^i\mathbf{v}, ...\}$. Give an example of a vector $\mathbf{v}$ for which the elements of the sequence approximate the zero vector, that is $\lim_{i\to\infty} A^i\mathbf{v} = \mathbf{0}$.

d) Is there a vector $\mathbf{v}$ for which the elements of the sequence $\{\mathbf{v}, A\mathbf{v}, A^2\mathbf{v_2}, ..., A^i\mathbf{v}, ...\}$ approximate the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, that is $\lim_{i\to\infty} A^i\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?

Problem 11: a) Let A and B be two matrices. Explain why N(AB) contains N(B)? Give an example when N(AB) is bigger than N(B).

b) Let $\mathbf{v_1}$ and $\mathbf{v_2}$ be two linear independent vectors in $\mathbf{R^3}$ and A the matrix with $\mathbf{v_1}$ and $\mathbf{v_2}$ as columns. Explain why $N(A^T A) = \{\mathbf{0}\}$.

c) For a general matrix A, what is the rank of $A^T A$?

d) For a general matrix A, explain why the normal linear system, that is $A^T A\mathbf{x} = A^T\mathbf{b}$, is always solvable?