The first 10 problems are worth 9 points each. The last two problems are 5 points each.

**Problem 1:** a) Find the A=LU factorization for

\[
A = \begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 3 & 1 \\
\end{bmatrix}
\]

b) Use the factorization to solve the linear system:

\[
\begin{align*}
x + y + 2z &= 3 \\
x + z &= 6 \\
2x + 3y + z &= 1
\end{align*}
\]

by braking the system \( Ax = b \) into two triangular systems.
Problem 2: Find the inverse of \( A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 5 \\ 2 & 5 & 1 \end{bmatrix} \) and use it to solve \( Ax = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \).
Problem 3: Find the complete solution to:

\[
\begin{bmatrix}
1 & 3 & 2 & 1 \\
2 & 6 & 4 & 1 \\
3 & 9 & 6 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
t \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
-1 \\
-3 \\
\end{bmatrix}
\]
Problem 4:  

a) Let \( V = \text{Span}( \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}) \).

a) Find a basis for \( V \).

b) What is the dimension of \( V \)?

c) Find 3 vectors, among the six vectors above, that are linearly dependent. Justify your answer.
Problem 5: Write down one matrix $A$ for each of the following conditions:

a) $A = A^T$ and $\det(A) = 6$

b) $\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$ is a basis for $C(A)$ and $\{\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\}$ is a basis for $N(A)$.

c) $C(A)$ is a subspace of $\mathbb{R}^4$ of dimension 2 and $\dim N(A) = 1$.

d) $A\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$.

e) $A$ is not diagonalizable but is invertible.
**Problem 6:** Find a basis for each of the following 3 subspaces:

a) The subspace of vectors \((b_1, b_2, b_3)\) in \(\mathbb{R}^3\) satisfying \(3b_1 = b_2\) and \(b_3 = b_1\).

b) The subspace of \(\mathbb{R}^4\) defined by \(x_1 + x_2 - 2x_3 + 3x_4 = 0\)

c) The subspace of all polynomials of degree 3 or less in \(x\) (this is a subspace of the vector space of all continuous functions \(f : \mathbb{R} \rightarrow \mathbb{R}\)).
Problem 7: Let $A = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ -1 & 1 \end{bmatrix}$.

a) Find the point $p$ in the column space $C(A)$ closest to $b = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

b) What is the best approximate solution to the linear system $Ax = b$ with $A$ and $b$ as above?

c) Find the projection matrix $P$ for the column space $C(A)$. 
Problem 8:  a) Using the cofactor formula find the determinant of

\[
A = \begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 0 & 1 & 1 \\
c & 0 & 2 & 1 \\
0 & c & 0 & 1
\end{bmatrix}
\]

b) For which \( c \) is \( A \) invertible? Explain.
Problem 9:  a) Find the general solution to the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= 4x + 3y \\
\frac{dy}{dt} &= -2x - 3y
\end{align*}
\]

b) Describe all initial conditions \((x(0), y(0))\) for which solutions satisfying those initial conditions are such that \(\lim_{t \to +\infty} (x(t), y(t)) = (0, 0)\).
Problem 10: Consider a car rental company which has 2 locations, one in Miami and the other in Orlando and has 1,000 cars. It is known that after each month 80% in Miami stay in Miami and that 10% of the cars in Orlando stay in Orlando.

a) In the long run how many cars will be in Miami?

b) If in the beginning of the year the distribution vector is \[ \begin{bmatrix} .3 \\ .7 \end{bmatrix} \], then what is the distribution vector at the end of the year?

Problem 11: and Problem 12: are left to your imagination