

# SAMPLE EXAM Math 210

12/10/02

You can choose any collection of 10 problems out of the first 12 problems. The last two problems are obligatory. In total you have to solve 12 problems. The last two problems are worth 7.5 points each, the other 10 are worth 8.5 points each.

**Problem 1:** Let  $\mathbf{v}_1 = (1, -1, 2)$ ,  $\mathbf{v}_2 = (2, 1, 3)$  and  $\mathbf{v}_3 = (1, -4, 3)$ .

- a) Show that  $\mathbf{v}_3$  is a linear combination of  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .
- b) Find a nonzero vector  $\mathbf{u}$  that is orthogonal to all the three vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$ .

**Problem 2:** Find the inverse of  $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 3 & 2 & 6 \end{bmatrix}$  and solve  $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

**Problem 3:** What is the A=LU factorization for  $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & -1 \\ 3 & 3 & 6 \end{bmatrix}$ .

**Problem 4:** Let  $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & c \\ 1 & 1 & 2 \end{bmatrix}$

- a) Describe the column space of  $A$  (line, plane or the entire 3-space) for the different values of  $c$ .
- b) Find a  $c$  and a  $\mathbf{b}$  for which the linear system  $A\mathbf{x} = \mathbf{b}$  has no solutions.

**Problem 5:** Write down a matrix  $A$  for each of the following conditions

- a) The linear system  $A\mathbf{x} = \mathbf{b}$ ,  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  has  $x_2$  and  $x_3$  as its free variables.
- b)  $A$  is  $3 \times 3$  and it has no zero entries but  $U = I$ .
- c) The only vector in  $N(A)$  is  $(0, 0, 0)$ .
- d) The vector  $(1, 1, 1)$  is both in  $C(A)$  and  $N(A)$ .

**Problem 6:** Find the complete solution to:

$$\begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

**Problem 7 :** Let  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$  and  $\mathbf{v}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

- a) Are the vectors  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  and  $\mathbf{v}_4$  linearly independent?
- b) Find two collections of 3 vectors among  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ ,  $\mathbf{v}_3$  and  $\mathbf{v}_4$  that form a basis of  $\mathbf{R}^3$ .
- c) What are the dimensions of  $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$  and  $W = \text{span}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ ?

**Problem 8:** If  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$ , find:

- a) The point  $\mathbf{p}$  in the column space  $C(A)$  closest to  $\mathbf{b} = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}$
- b) The projection matrix  $P$  for the column space  $C(A)$ .
- c) The orthogonal complement of the column space  $C(A)$ .

- Problem 9:** a) Find orthonormal vectors  $\mathbf{u}_1$  and  $\mathbf{u}_2$  in the plane spanned by  $\mathbf{v}_1 = (1, -1, 2)$ ,  $\mathbf{v}_2 = (2, 1, 3)$ .
- b) Find a vector  $\mathbf{u}_3$  such that  $\mathbf{u}_1$ ,  $\mathbf{u}_2$  and  $\mathbf{u}_3$  form an orthonormal basis of  $\mathbf{R}^3$  (i.e. a basis where the vectors are orthonormal).

**Problem 10:** Using the cofactor formula find the determinant of

$$A = \begin{bmatrix} 2 & 0 & 4 & -1 \\ 1 & -2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 4 & -2 & 0 \end{bmatrix}$$

**Problem 11:** Diagonalize  $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$  if possible. If it is not possible find a Jordan matrix that is similar to  $A$ .

**Problem 12:** a) Find the general solution to the system of differential equations

$$\begin{aligned}\frac{dx}{dt} &= 2x + 3y \\ \frac{dy}{dt} &= -x - 2y\end{aligned}$$

b) Write a particular solution  $(x(t), y(t))$  of the system of differential equations described above such that  $\lim_{t \rightarrow \infty} x(t) = 0$  and  $\lim_{t \rightarrow \infty} y(t) = 0$ .

c) Can we have a solution  $(x(t), y(t))$  of the system of differential equations described above such that  $\lim_{t \rightarrow \infty} x(t) = 0$  but  $\lim_{t \rightarrow \infty} y(t) = \infty$ .

**Problem 13:** Let  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

- a) Find the eigenvalues of  $A$ . (Hint: do not use  $\det(A-\lambda I)=0$  to find the eigenvalues)
- b) Is  $A$  diagonalizable?
- c) Does  $A^k \rightarrow 0$  as  $k \rightarrow \infty$ ?
- d) Is the matrix  $A-9I$  singular?

**Problem 14:** a) If  $A$  is a  $3 \times 3$  matrix such that  $A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $A \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$  and

$A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ . What is the  $\lim_{k \rightarrow \infty} A^k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ ?

b) Suppose  $A$  is a  $3 \times 3$  matrix and  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  are 3 vectors in  $\mathbf{R}^3$  such that  $A\mathbf{v}_1 = 3\mathbf{v}_1$ ,  $A\mathbf{v}_2 = 2\mathbf{v}_2$  and  $A\mathbf{v}_3 = \mathbf{v}_3$ . What is the determinant of  $A$ ? Do  $\mathbf{v}_1$ ,  $\mathbf{v}_2$  and  $\mathbf{v}_3$  form a basis of  $\mathbf{R}^3$ ? Explain your answers.

c) Suppose the linear system

$$\begin{aligned} a_1x + b_1y + c_1z &= 0 \\ a_2x + b_2y + c_2z &= 0 \end{aligned}$$

defines a line in  $\mathbf{R}^3$ . Can we form a basis for  $\mathbf{R}^2$  using two vectors among  $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  and  $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ ? Explain your answer